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## Beginning Algebra

by

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## Chapter 1

## Introduction to Algebra

### 1.1 YouTube

http://www.youtube.com/playlist?list=PL37F70081BDEFEBA9\&feature=view_all
A vista into history from
www.ucs.louisiana.edu/ sxw8045/history.htm
In the 7 th and 8 th centuries the Arabs, united by Mohammed, conquered the land from India, across northern Africa, to Spain. In the following centuries (through the 14th) they pursued the arts and sciences and were responsible for most of the scientific advances made in the west. Although the language was Arabic many of the scholars were Greeks, Christians, Persians, or Jews. Their most valuable contribution was the preservation of Greek learning through the middle ages, and it is through their translations that much of what we know today about the Greeks became available. In addition they made original contributions of their own.

They took over and improved the Hindu number symbols and the idea of positional notation. These numerals (the Hindu-Arabic system of numeration) and the algorithms for operating with them were transmitted to Europe around 1200 and are in use throughout the world today.

Like the Hindus, the Arabs worked freely with irrationals. However they took a backward step in rejecting negative numbers in spite of having learned of them from the Hindus.

In algebra the Arabs contributed first of all the name. The word "algebra" come from the title of a text book in the subject, Hisab al-jabr w'al muqabala, written about 830 by the astronomer/mathematician Mohammed ibn-Musa al-Khowarizmi. This title is sometimes translated as "Restoring and Simplification" or as "Transposition and Cancellation." Our word "algorithm" in a corruption of al-Khowarizmi's name.

The algebra of the Arabs was entirely rhetorical.

They could solve quadratic equations, recognizing two solutions, possibly irrational, but usually rejected negative solutions. The poet/mathematician Omar Khayyam (1050-1130) made significant contributions to the solution of cubic equations by geometric methods involving the intersection of conics.

Like Diophantus and the Hindus, the Arabs also worked with indeterminate equations.
Here are other links, if you are interested:
www.algebra.com/algebra/about/history
www.helpalgebra.com/info/algebrahistory.htm
www.freemathhelp.com/history-algebra.html aleph0.clarku.edu/djoyce/mathhist/algebra.html www.ehow.com/video_4977241_ who-invented-algebra.html
mathforum.org/sum95/ruth/history.html

Think of Algebra basically as Arithmetic using unknown numbers, variables. The variable $\boldsymbol{x}$ stands for an unknown, whereas $\mathbf{3}$ is understood to be three. $\boldsymbol{x}$ is a locked $\boldsymbol{x}$-box which contains a number.

It is important to write algebraic expressions neatly.
Bad example 1:
$4+5$

$$
+6
$$

7
Did you mean
$\frac{4+5+6}{7} \quad$ or $\quad \frac{4+5}{7}+6 \quad$ or $\quad \frac{4}{7}+5+6 \quad ?$
Example 2:
6/2•4 which is interpreted as $3 \cdot 4=12$
Did you mean
$\frac{6}{2} \cdot 4 \quad$ or $\quad \frac{6}{2 \cdot 4}$
Example 3:
$\sqrt{144 .} 25$

Did you mean
$\sqrt{144 \cdot 25}$ or $\sqrt{144} \cdot 25$
Some vocabulary:
Variable: A symbol (Letter like $\boldsymbol{x}$ ) representing a number.
Algebraic Expression: A collection of symbols (other than $=$ ), like $2 x+3, \frac{x-3}{\sqrt{\pi}}$
Equation: A mathematical sentence relating two expression with an $=$ sign.
For homework:
Study each example in the text by writing it out on paper. Get to the point where you can reproduce the example without referring to any source. Then you will be ready to start the homework. While doing homework, do not copy an example statement that you just read from an example with new numbers from the homework problem. You cannot consult examples from the text when you take an exam.

Memorize vocabulary. Algebra (like all mathematics) is a foreign language that people use all over the world.

Blank page for student note taking.

## Chapter 2

## The Commutative, Associative, and Distributive Laws/Properties

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### 2.1 YouTube

http://www.youtube.com/playlist?list=PL42A47D20EB25677D

### 2.2 The Commutative Property of Addition and Multiplication



2 commuted (traveled) from position 1 to position 2
and
3 commuted (traveled) from position 2 to position 1.
(A commuter is a traveler. Do not say "communitive" property)
The picture illustrates the commutative property of addition.

6 CHAPTER 2. THE COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS/PROPERTIES

In general

$$
a+b=b+a
$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any real numbers $\left(\right.$ like $\left.-\mathbf{6 . 4}, \frac{\mathbf{2}}{\mathbf{7}}, \boldsymbol{\pi}\right)$. Any real number can hide in the $\boldsymbol{a}$-box or the $b$-box.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$

Is subtraction commutative?
Is $\mathbf{3 - 1}=\mathbf{1 - 3}$ a true statement?
No, because $\mathbf{3 - 1}-\mathbf{2}$ and $\mathbf{1 - 3}=\mathbf{- 2}$. If we find one counterexample, one example that shows that subtraction is not commutative, the general property (using $\boldsymbol{a}$ and $\boldsymbol{b}$ ) does not exist.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$


2 commuted (traveled) from position 1 to position 2
and
3 commuted (traveled) from position 2 to position 1.
The picture illustrates the commutative property of multiplication.
In general

$$
a b=b a
$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are any real numbers $\left(\right.$ like $\left.-\mathbf{6 . 4}, \frac{\mathbf{2}}{\mathbf{7}}, \boldsymbol{\pi}\right)$ ). Any real number can hide in the $\boldsymbol{a}$-box or the $b$-box.

Note that $\boldsymbol{a} \boldsymbol{b}$ means $\boldsymbol{a}$ times $\boldsymbol{b}$.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$
Is division commutative?
Is $4 \div 2=2 \div 4$ a true statement?

No, because $\mathbf{4 \div 2 = 2}$ and $\mathbf{2} \div \mathbf{4}=\frac{\mathbf{1}}{\mathbf{2}}=\mathbf{0 . 5}$. If we find one counterexample, one example that shows that division is not commutative, the general property (using $\boldsymbol{a}$ and $\boldsymbol{b}$ ) does not exist.

### 2.3 The Associative Property of Addition and Multiplication

$$
5+\underbrace{(4 \oplus 3)}=\underbrace{(5+4)} \oplus 3
$$

$$
4 \text { is associated (grouped) } \quad 4 \text { is associated (grouped) }
$$

$$
\text { with } 3
$$

$$
\text { with } 5
$$

$5+(4+3)=5+7=12$ and $(5+4)+3=9+3=12$ also.
The picture illustrates the Associative property of addition.
In general

$$
a+(b+c)=(a+b)+c
$$

where $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are any real numbers.
Is subtraction associative?
Is $5-(4-3)=(5-4)-3$ a true statement?
No, because
$5-(4-3)=5-1=4$ and $(5-4)-3=1-3=-2$.
The general property (using $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ ) does not exist.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$

$$
5 \cdot \underbrace{(4 \Im 3)}=\underbrace{(5 \bigodot 4)} \cdot 3
$$

| 4 is associated (grouped) with 3 | 4 is associated (groupe with 5 |
| :---: | :---: |
| $5(4 \cdot 3)=5(12)=60$ an | $=(20) 3=60$ also |

The picture illustrates the Associative Property of Multiplication.

8 CHAPTER 2. THE COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS/PROPERTIES

In general

$$
a(b c)=(a b) c
$$

where $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are any real numbers.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$

Is division associative?
Is $16 \div(4 \div 2)=(16 \div 4) \div 2$ a true statement?
No, because
$16 \div(4 \div 2)=16 \div 2=16 \div 2=8$
and
$(16 \div 4) \div 2=(4) \div 2=2$.
Division is not associative, the general property (using $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ ) does not exist.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$

### 2.4 The Distributive property of Multiplication over Addition and/or Subtraction


$5(4+3)=5(7)=35$
and
$(5)(4)+(5)(3)=20+15=35$
also.
The picture illustrates the Distributive property of multiplication over addition.
In general

$$
a(b+c)=a b+a c
$$

where $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are any real numbers.

$5(4-3)=5(1)=5$ and $(5)(4)-(5)(3)=20-15=5$ also.
The picture illustrates the Distributive property of multiplication over Subtraction.
In general

$$
a(b+c)=a b+a c
$$

where $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ are any real numbers.

$5(4 \cdot 3)=5(12)=60$
and
$(5)(4) \cdot(5)(3)=20 \cdot 15=300$
which is different.
The example shows that the distributive property does not apply to multiplication over multiplication.

$16 \div(4 \div 2)=16 \div 2=8$
and
$(16 \div 4) \div(16 \div 2)=4 \div 8=\frac{1}{2}$
which is different.
The example shows that the distributive property does not apply to division over division.

### 2.5 Factoring the GCF

The distributive property of multiplication over addition/subtraction can be applied in reverse.
$a(b \pm c)=a b \pm a c$ implies $a b \pm a c=a(b \pm c)$.
FDactoring is the art of taking a sum (addition of terms) or difference (subtraction of terms) into a product (multiplication of factors).

Example :
Factor $15 x+20 y$.

## Solution:

$$
\begin{aligned}
15 x+20 y & =3 \cdot 5 x+4 \cdot 5 y \\
& =5(3 x+4 y)
\end{aligned}
$$

### 2.6 Exercise 2

State the property or indicate that no property is applicable.

1. $5+x=x+5$
2. $7(a-2)=7 a-7(2)$
3. $5(x \cdot 2)=(5 x)(2)$
4. $(y+2)+4=4+(y+2)$
5. $6 \cdot(t \cdot 2)=(6 t) \cdot(6 \cdot 2)$
6. $\boldsymbol{w}-5=5-\boldsymbol{w}$
7. $x y=y x$
8. $a \pi=\pi a$
9. $(-4.7)+[(6.5)+(\sqrt{2})]=[(-4.7)+(6.5)]+(\sqrt{2})$
10. $x-4=4-x$
11. Factor $\mathbf{1 2 x}+\mathbf{1 8}$
12. Factor $24 x^{2}+\mathbf{1 2 x}$

## $S T O P!$

The solutions follow:
Keep them covered up till you have worked out each of the problems above.

1. $\mathbf{5}+\boldsymbol{x}=\boldsymbol{x}+\mathbf{5}$

Solution:
$\mathbf{5}+\boldsymbol{x}=\boldsymbol{x}+\mathbf{5}$ illustrates the commutative property of addition. The terms commuted to each other's position.
2. $7(a-2)=7 a-7(2)$

Solution:
$7(a-2)=7 a-7(2)$ illustrates the distributive property of multiplication over addition.
3. $5(x \cdot 2)=(5 x)(2)$

Solution:
$\mathbf{5 ( x \cdot 2 )}=(5 \boldsymbol{x})(2)$ illustrates the associative property of multiplication.
4. $(y+2)+4=4+(y+2)$

Solution:
$(\boldsymbol{y}+2)+4=4+(\boldsymbol{y}+2)$ illustrates the commutative property of addition. $(\boldsymbol{y}+2)$ and 4 have commuted to each other's position. $\boldsymbol{y}$ remains grouped with 2.
5. $6 \cdot(t \cdot 2)=(6 t) \cdot(6 \cdot 2)$

## Solution:

$\mathbf{6} \cdot(\boldsymbol{t} \cdot \mathbf{2})=(\mathbf{6 t}) \cdot(\mathbf{6} \cdot \mathbf{2})$ does not illustrate any of the mentioned properties. You cannot distribute multiplication over multiplication.
6. $\boldsymbol{w}-\mathbf{5}=\mathbf{5}-\boldsymbol{w}$

Solution:
$\boldsymbol{w}-\mathbf{5}=\mathbf{5}-\boldsymbol{w}$ does not illustrate any of the mentioned properties. Subtraction is not commutative.
We shall learn later that $\boldsymbol{w}-\mathbf{5}=\boldsymbol{w}+(\mathbf{5})=(-\mathbf{5})+\boldsymbol{w}$ because addition is commutative.
7. $x y=y x$

Solution:
$\boldsymbol{x} \boldsymbol{y}=\boldsymbol{y} \boldsymbol{x}$ illustrates the commutative property of multiplication. The terms commuted to each other's position.
8. $a \div \pi=\pi \div a$

Solution:
$\boldsymbol{a} \div \boldsymbol{\pi}=\boldsymbol{\pi} \div \boldsymbol{a}$ does not illustrate any of the mentioned properties. Division is not commutative.
9. $(-4.7)+[(6.5)+(\sqrt{2})]=[(-4.7)+(6.5)]+(\sqrt{2})$

## Solution:

$(-4.7)+[(6.5)+(\sqrt{2})]=[(-4.7)+(6.5)]+(\sqrt{2})$ illustrates the associative property of addition. 6.5 is associated (grouped) with $\sqrt{2}$ and $\mathbf{- 4 . 7}$.
10. $x-4=4-x$

Solution:
$\boldsymbol{x}-\mathbf{4}=\mathbf{4}-\boldsymbol{x}$ does not illustrate any of the mentioned properties. Subtraction is not commutative.
11. Factor $12 x+18$ Solution:
$12 x+18=(3)(4) x+(3)(6)$

$$
=3(4 x+6)
$$

A quick check: $3(4 x+6)=3(4 x)+3(6)=12 x+18$
12. Factor $24 x^{2}+12 x$ Solution:

$$
\begin{aligned}
24 x^{2}+12 x & =2(12) x x+1(12) x \\
& =12 x(2 x+1)
\end{aligned}
$$

A quick check: $12 x(2 x+1)=12 x(2 x)+12 x(1)=24 x^{2}+12 x$

## Chapter 3

## Additional Properties of Real Numbers

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### 3.1 YouTube

https://www.youtube.com/playlist?list=PL04BA33DF9D1B5167\&feature=view_all

### 3.2 Identity for Addition

$\mathbf{0}$ is the identity element for addition because $\mathbf{0}+\mathbf{5}=\mathbf{5}$ AND $\mathbf{5}+\mathbf{0}=\mathbf{5}$. In general

$$
0+a=a \text { AND } a+0=a
$$

where $\boldsymbol{a}$ is any real number.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *$

Is $\mathbf{0}$ the identity element for subtraction?
Is $\mathbf{5 - 0}=\mathbf{5}$ a true statement? Yes, but $\mathbf{0 - 5} \neq \mathbf{5}$.
Therefore $\mathbf{0}$ is not the identity element for subtraction.
$\mathbf{0}$ has an interesting property for multiplication. $\mathbf{0} \cdot \boldsymbol{a}=\mathbf{0}$ and $\boldsymbol{a} \cdot \mathbf{0}=\mathbf{0}$ for all real numbers $\boldsymbol{a}$.

### 3.3 Identity for Multiplication

$\mathbf{1}$ is the identity element for multiplication because $\mathbf{1} \cdot \mathbf{5}=\mathbf{5}$ AND $\mathbf{5} \cdot \mathbf{1}=\mathbf{5}$. In general

$$
(1) a=a \text { AND } a(1)=a
$$

where $\boldsymbol{a}$ is any real number.
$* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * ~$

Is $\mathbf{1}$ the identity element for division?
Is $\mathbf{5} \div \mathbf{1}=\mathbf{5}$ a true statement? Yes, but $\mathbf{1} \div \mathbf{5} \neq \mathbf{5}$.
Therefore $\mathbf{1}$ is not the identity element for division.
$\mathbf{0}$ has an interesting property for division. $\boldsymbol{a} \div \mathbf{0}$ is not defined for all real numbers $\boldsymbol{a}$.
Here is one explanation:
$12 \div 3=\frac{12}{3}=4$ because $(4)(3)=12$.
Similarly
$a \div b=\frac{a}{b}=c$ because $(b)(c)=a$.
Now
$\mathbf{1 2} \div \mathbf{0}=\frac{\mathbf{1 2}}{\mathbf{0}}=\boldsymbol{x}$ because $(\mathbf{0})(\boldsymbol{x})=\mathbf{1 2}$. But $\mathbf{0} \boldsymbol{x}=\mathbf{0}$, not 12 . No product $\mathbf{0} \cdot \boldsymbol{x}$ results in the dividend (numerator) 12 .
$\boldsymbol{a} \div 0=\frac{\boldsymbol{a}}{\mathbf{0}}=\boldsymbol{c}$ because $(0)(c)=\boldsymbol{a}$ is impossible.

### 3.4 Opposites (Additive Inverses) and Reciprocals (Multiplicative Inverses)

What number added to 5 results in the identity for addition?
$5+(-5)=0$
In general, $\boldsymbol{a}+(\boldsymbol{a})=\mathbf{0} . \mathbf{- 5}$ is called the opposite (or additive inverse) of $\mathbf{5}$. Every real number has a unique opposite except for $\mathbf{0}$ which is its own opposite.

Same distance, opposite sides of $\mathbf{0}$


Example 1:
What is the opposite of $\mathbf{- 3}$ ?

## Solution:

The opposite of $-\mathbf{3}$ is $\mathbf{3}$. Thus $-(-\mathbf{3})=\mathbf{3}$.

What number multiplied by 5 results in the identity for multiplication?
$5 \cdot \frac{1}{5}=1$
In general, $\boldsymbol{a} \cdot \frac{\mathbf{1}}{\boldsymbol{a}}=\mathbf{1} . \quad \frac{\mathbf{1}}{\mathbf{5}}$ is called the reciprocal
(or multiplicative inverse) of $\mathbf{5}$. Every real number has a unique opposite except for $\mathbf{0}$ because division by $\mathbf{0}$ is undefined.

### 3.5 Absolute Value

The absolute value of a number is the measure of its distance from $\mathbf{0}$ (to the left or to the right).
-6 is $\mathbf{6}$ units away from $\mathbf{0}$. $\mathbf{3}$ is $\mathbf{3}$ units away from $\mathbf{0}$.


The absolute value of $\mathbf{- 6}$, written $|-\mathbf{6}|$ is $\mathbf{6}$.
The absolute value of $\mathbf{3}$, written $|\mathbf{3}|$ is $\mathbf{3}$.
You were probably taught to copy the number between the absolute value bars and erase the negative sign, if any. That probably served its purpose, but it is insufficient. The formal definition of absolute value is

$$
|\boldsymbol{x}|=\left\{\begin{array}{rll}
\boldsymbol{x} & \text { if } & \boldsymbol{0} \leq \boldsymbol{x} \\
-\boldsymbol{x} & \text { if } & \boldsymbol{x}<\mathbf{0}
\end{array}\right.
$$

## Example 2:

Use the definition of absolute value to find $|-\mathbf{6}|$.

## Solution:

$|-6|=\left\{\begin{array}{c}\cdots \\ -(-6)=6 \quad \text { since } \quad-6 \leq 0\end{array}\right.$

## Example 3:

Use the definition of absolute value to find $|\mathbf{3}|$.

## Solution:

$$
|3|=\left\{\begin{array}{c}
\mathbf{3} \\
\cdots
\end{array} \text { since } 0 \leq 3\right.
$$

## Example 4:

Evaluate $-|\boldsymbol{x}|$ if $\boldsymbol{x}=-\mathbf{6}$

## Solution:

$$
\begin{aligned}
-|x| & =-|-6| & & \text { Substitute }-6 \text { for } \boldsymbol{x} . \\
& =-6 & & \text { since }|-6|=6
\end{aligned}
$$

Do not confuse $-(-6)=\mathbf{6}$ with $-|-6|=-6$.

### 3.6 Inequality Symbol

There can only be one true symbol relating a pair of real numbers (trichotomy).
$\boldsymbol{a}<\boldsymbol{b}, \boldsymbol{a}=\boldsymbol{b}$, or $\boldsymbol{a}>\boldsymbol{b}$.
Place $\boldsymbol{a}$ and $\boldsymbol{b}$ on a number line. The number to the the right is the larger one. For example $\mathbf{- 6}<\mathbf{- 1}$ or $-1>-6 .-6 \leq-1$ or $-1 \geq-6$ would also be correct.


## Example 5:

Insert $<,=$, or $>$ between $\mathbf{- 6}$ $\qquad$ $-3$.

## Solution:

$-\mathbf{6}<-\mathbf{3}$ because $-\mathbf{3}$ is to the right of $\mathbf{- 6}$.



The stronger (bigger) number holds the handle of the knife, while the smaller (weaker) number gets the point of the blade.

Do you have trouble remembering which side of $<$ is the larger number?
Use of $\neq$ :
If $\boldsymbol{a} \neq \boldsymbol{b}$ then $\boldsymbol{a}<\boldsymbol{b}$ or $\boldsymbol{a}>\boldsymbol{b}$. Thus $-\mathbf{6} \neq \mathbf{3}$ and $\mathbf{3} \neq \mathbf{1}$.
Use of $\leq$ :
$\boldsymbol{a} \leq \boldsymbol{b}$ is true if either $\boldsymbol{a}<\boldsymbol{b}$ or $\boldsymbol{a}=\boldsymbol{b}$.
Thus $\mathbf{2} \leq \mathbf{3}$ is true because $\mathbf{2}<\mathbf{3}$ and
$\mathbf{2} \leq 2$ is true because $2=2$ but
$\mathbf{2} \leq \mathbf{1}$ is false because neither $\mathbf{2}<\mathbf{1}$ nor $\mathbf{2}=\mathbf{1}$ is true.

## Example 6:

Explain why $\boldsymbol{a}<\boldsymbol{b}$ is true if $\boldsymbol{a}<\mathbf{0}$ and $\mathbf{0}<\boldsymbol{b}$.

## solution:

On a number line, $\boldsymbol{a}$ is to the left of $\mathbf{0}$ while $\boldsymbol{b}$ is to the right. Therefore $\boldsymbol{a}$ is always to the left of $\boldsymbol{b}$ resulting in $\boldsymbol{a}<\boldsymbol{b}$.

## Example 7:

Find two numbers such that $|\boldsymbol{x}|=\mathbf{2 . 5}$.

## Solution:

We are looking for two numbers whose distance from $\mathbf{0}$ is $\mathbf{2 . 5}$ units. The numbers are $\boldsymbol{x}=\mathbf{- 2 . 5}$ and $x=2.5$.

## Example 8:

Write the following statement mathematically:
"Mr. Baba's salary is at least \$20,000". Use the variable $\boldsymbol{x}$ for Mr. Baba's salary.
Solution:
"At least" means a certain number or more. Thus $\boldsymbol{x} \geq \mathbf{\$ 2 0}, \mathbf{0 0 0}$.

## Example 9:

Write the following statement mathematically:
"Mr. Baba's salary is more than \$20,000". Use the variable $\boldsymbol{x}$ for Mr. Baba's salary.

## Solution:

"More than" means $>$. Thus $\boldsymbol{x}>\mathbf{\$ 2 0 , 0 0 0}$.

## Example 10:

Write the following statement mathematically:
"Mr. Baba's salary is at most \$20,000". Use the variable $\boldsymbol{x}$ for Mr. Baba's salary.

## Solution:

"At most" means a certain number or less. Thus $\boldsymbol{x} \leq \mathbf{\$ 2 0}, \mathbf{0 0 0}$.

## Example 11:

Write the following statement mathematically:
"Mr. Baba's salary is less than \$20,000". Use the variable $\boldsymbol{x}$ for Mr. Baba's salary.

## Solution:

"Less than" means $<$. Thus $\boldsymbol{x}<\mathbf{\$ 2 0 , 0 0 0}$.

### 3.7 Rounding

## Example 12:

Round 14, 371 to the nearest hundred.

## Solution:

14371
14371 Circle the hundred digit " 3 "
14300 Substitute 0 for digits to the right of the circle.
The first dropped digit, 7 , was $\geq \mathbf{5}$,
so the circled digit is incremented.
$\mathbf{1 4}, \mathbf{3 7 1}$ rounded to the nearest hundred is $\mathbf{1 4}, 400$


Note that $\mathbf{1 4}, \mathbf{3 7 1}$ is in the second half of the number line segments above. The number is closer to 14,400 than 14,300 .

Example 13:

Round 14,371 to the nearest thousand.

## Solution:

## 14371

1(4)371 Circle the thousand digit " 4 "
14) 000 Substitute 0 for digits to the right of the circle.

The first dropped digit, 3 , was $<\mathbf{5}$,
so the circled digit remains unchanged.
14,371 rounded to the nearest thousand is 14,000


Note that $\mathbf{1 4}, \mathbf{3 7 1}$ is in the first half of the number line segments above. The number is closer to 14,000 than 15,000 .

Note: If the number to be rounded (say 14,500) falls exactly between the extremes $(14,000$ and 15,000$)$ always round up to the right extreme $(15,000)$.

### 3.8 Exercise 3

Justify the following statements or state why the statement is false.

1. $5+(-5)=0$
2. $\left(\frac{4}{9}\right)\left(\frac{9}{4}\right)=1$
3. $(-2.8)(1)=-2.8$
4. $(-2.8)(0)=0$
5. $(-2.8)+(0)=-2.8$
6. $6(7 \cdot 8)=(7 \cdot 8) 6$
7. Every real number has a reciprocal.
8. The sum of the absolute value of any number $\boldsymbol{x}$ and its opposite is $\boldsymbol{2} \boldsymbol{x}$.
9. Find two numbers equal to their reciprocals.
10. The sum of a number and its opposite is 0 .
11. The product of a number and its reciprocal is 1 .
12. $\mathbf{0}$ is the identity for subtraction because $\boldsymbol{x}-\mathbf{0}=\boldsymbol{x}$.
13. $\mathbf{1}$ is the identity for division because $\frac{\boldsymbol{x}}{\mathbf{1}}=\boldsymbol{x}$.
14. Find two numbers such that $|\boldsymbol{x}|=\mathbf{- 6}$.
15. Find two numbers such that $|x|=6$.
16. Find two numbers such that $|\boldsymbol{x}|>6$.
17. Is $\mathbf{- 1} \leq \mathbf{- 5}$ true or false?.
18. There are at most 270 calories in a consumer product. What is the highest number of calories in the product? (Use a whole number.)
19. According to kidshealth.org, most full-term babies weigh at least 6 lbs 2 oz. According to this data, what is the highest weight a full-term baby can weigh? (Assume "most babies" in the statement is replaced by "all babies".) (See
(www.ehow.com/facts_5349623_normal-birth-weight-length.html)
20. $\boldsymbol{x} \geq \boldsymbol{x}$ is a true statement. Correct?
21. Round 23.4567 to the nearest hundredth.

## STOP!

The solutions follow:
Keep them covered up till you have worked out each of the problems above.

1. $5+(-5)=0$

## Solution:

$5+(-5)=0$ because 5 and $-\mathbf{5}$ are opposites.
2. $\left(\frac{4}{9}\right)\left(\frac{9}{4}\right)=1$

## Solution:

$\left(\frac{\mathbf{4}}{\mathbf{9}}\right)$ and $\left(\frac{\mathbf{9}}{\mathbf{4}}\right)=1$ are reciprocals.
3. $(-2.8)(1)=-2.8$

## Solution:

$(-2.8)(\mathbf{1})=\mathbf{- 2 . 8}$ because $\mathbf{1}$ is the identity for addition.
4. $(-2.8)(0)=0$

## Solution:

$$
(-\mathbf{2 . 8})(\mathbf{0})=\mathbf{0} \text { Any number times } 0 \text { is } 0 \text { (Multiplication property of } \mathbf{0})
$$

5. $(-2.8)+(0)=-2.8$

## Solution:

$(-2.8)+(\mathbf{0})=-\mathbf{2 . 8}$ The sum of a number and 0 is that number. $\mathbf{0}$ is the identity for addition.
6. $6(7 \cdot 8)=(7 \cdot 8) 6$

## Solution:

$6(7 \cdot 8)=(7 \cdot 8) 6$ Multiplication is commutative. $7 \cdot 8$ and 6 have commuted to each other's position.
7. Every real number has a reciprocal.

## Solution:

Every real number has a reciprocal. This statement is false. $\mathbf{0}$ does not have a reciprocal $\frac{\mathbf{1}}{\mathbf{0}}$ because division by $\mathbf{0}$ is undefined. Do not say 0 is undefined. If you score 0 on a test you are very well aware of its meaning.
8. The sum of the absolute value of any number $\boldsymbol{x}$ and its opposite is $\mathbf{2 \boldsymbol { x }}$.

## Solution:

The sum of the absolute value of any number $\boldsymbol{x}$ and its opposite is $\mathbf{2 x}$.
The statement is false if $\boldsymbol{x} \geq \mathbf{0}$ as illustrated by the following:
Let $\boldsymbol{x}=\mathbf{7}$. The opposite of $\boldsymbol{x}$ is -7 . The sum of $|\boldsymbol{x}|=|\boldsymbol{7}|=\mathbf{7}$ and its opposite is
$-x=-(7)=-7$ is $7+(-7)=0$,
not $2 x=(2)(7)=14$.
The statement is true if $\boldsymbol{x}<\mathbf{0}$ :
Let $\boldsymbol{x}=-\mathbf{3}$. The opposite of $\boldsymbol{x}$ is $-(-3)=\mathbf{3}$. The sum of $|\boldsymbol{x}|=|-3|=\mathbf{3}$ and its opposite is $-x=-(-3)=3$ is $3+(3)=2(3)=6$.
9. Find two numbers equal to their reciprocals.

## Solution:

Find two numbers is equal to their reciprocals.
If $x=1$ its reciprocal is $\frac{1}{1}=1$.
If $\boldsymbol{x}=-1$ its reciprocal is $\frac{1}{-1}=-\mathbf{1}$.
10. The sum of a number and its opposite is 0 .

## Solution:

The sum of a number and its opposite is $0 . \boldsymbol{x}+(-\boldsymbol{x})=\mathbf{0}$.
Let $\boldsymbol{x}=4$ then $-\boldsymbol{x}=-4$ is its opposite. The sum is $\boldsymbol{x}+(-\boldsymbol{x})=4+(-4)=0$.
Let $\boldsymbol{x}=\mathbf{- 3}$ then $-\boldsymbol{x}=-(-\mathbf{3})=\mathbf{3}$ is its opposite. The sum is $\boldsymbol{x}+(-\boldsymbol{x})=-\mathbf{3}+(\mathbf{3})=\mathbf{0}$.
11. The product of a number and its reciprocal is 1 .

## Solution:

The product of a number and its reciprocal is 1 if $\boldsymbol{x} \neq \mathbf{0} . \boldsymbol{x}\left(\frac{\mathbf{1}}{\boldsymbol{x}}\right)=\mathbf{1}$.
If $\boldsymbol{x}=\mathbf{0}$, the statement is false because $\mathbf{0}$ does not have a reciprocal.
12. $\mathbf{0}$ is the identity for subtraction because $\boldsymbol{x}-\mathbf{0}=\boldsymbol{x}$.

## Solution:

$\mathbf{0}$ is the identity for subtraction because $\boldsymbol{x}-\mathbf{0}=\boldsymbol{x}$. The statement is false because $\mathbf{0}-\boldsymbol{x}=-\boldsymbol{x}$, not $\boldsymbol{x}$.

Let $\boldsymbol{x}=\mathbf{3} . \mathbf{3}-\mathbf{0}=\mathbf{3}$ but $\mathbf{0}-\mathbf{3}=-\mathbf{3} \neq \mathbf{3}$.
13. $\mathbf{1}$ is the identity for division because $\frac{\boldsymbol{x}}{\mathbf{1}}=\boldsymbol{x}$.

## Solution:

1 is not the identity for division because even though
$\frac{\boldsymbol{x}}{\mathbf{1}}=\boldsymbol{x}, \frac{\mathbf{1}}{\boldsymbol{x}} \neq \boldsymbol{x}$.
14. Find two numbers such that $|\boldsymbol{x}|=\mathbf{- 6}$.

## Solution:

Find two numbers such that $|\boldsymbol{x}|=\mathbf{- 6}$. You cannot find any numbers because absolute value is never negative. This problem has no solution.
15. Find two numbers such that $|\boldsymbol{x}|=\mathbf{6}$.

## Solution:

Find two numbers such that $|\boldsymbol{x}|=\mathbf{6}$.


The numbers are $\boldsymbol{x}=-\mathbf{6}$ and $\boldsymbol{x}=\mathbf{6}$.
16. Find two numbers such that $|\boldsymbol{x}|>6$.

## Solution:

Find two numbers such that $|\boldsymbol{x}|>\mathbf{6}$. There are infinitely many solutions, like $\boldsymbol{x}=\mathbf{7}, \boldsymbol{x}=\mathbf{6}, \mathbf{5 4 3}$, $x=6.03, x=-6.5, x=-27$.

17. Is $-\mathbf{1} \leq-\mathbf{5}$ true or false?.

## Solution:

$-1 \leq-5$ is false because $-1<-5$ is false and $-1=-5$ is false.

18. There are at most 270 calories in a consumer product. What is the highest number of calories in the product? (Use a whole number.)

## Solution:

The highest number is 270 calories. "at most" means "up to" and including.
19. Most full-term babies weigh at least 6 lbs 2 oz . According to this data, what is the highest weight a full-term baby can weigh?

## Solution:

"at least" means "that much or more". There is no largest number. The problem has no solution according to the given data.
Of course there is a largest number ( 9 lbs .2 oz . according to the website), but it is not given in the statement of the problem.
20. $\boldsymbol{x} \geq \boldsymbol{x}$ is a true statement. Correct?

## Solution:

$\boldsymbol{x} \geq \boldsymbol{x}$ is indeed a true statement because $\boldsymbol{x}=\boldsymbol{x}$ for all real numbers.
21. Round 23.4567 to the nearest hundredth.

## Solution:

23.4567
23.4(5)67 Circle the hundredth digit " 5 "
23.4500 Substitute 0 for digits to the right of the circle.
23.46 The first dropped digit, 6 , is $\geq \mathbf{5}$.

The circled digit is incremented.
$\mathbf{2 3 . 4 5 6 7}$ rounded to the nearest hundredth is $\mathbf{2 3 . 4 6}$


Note that 23.4567 is in the second half of the number line segments above. The number is closer to 123.46 than 23.45 .

Blank page for student note taking.

## Chapter 4

## Arithmetic of Signed Numbers

### 4.1 YouTube

### 4.2 Real Numbers Overview

## Real Numbers



Also known as Counting Number or as Positive Integerss


Why does the number of rational and the number of irrational numbers appear to be the same on the number line above?

A rational number (ratio of two integers) can be rewritten as a terminating decimal (like $\frac{\mathbf{5}}{\mathbf{2}}=\mathbf{2 . 5}$ ) or a decimal with infinitely many decimals in repeating groups (like $\frac{\mathbf{5}}{\mathbf{1 1}}=\mathbf{0 . 4 5 4 5 4 5 4 5 4 5 4 5 4 5} \ldots$ )

An irrational number can be represented as a decimal number with infinitely many decimals, but no group of digits repeats itself.

Can we construct an irrational number?
Start with a decimal, like $\mathbf{0 . 2}$. Append a different group of digits, like $\mathbf{1 2}$ to get $\mathbf{0 . 2 1 2}$. Then append a different group, like $\mathbf{1 1 2}$ to get $\mathbf{0 . 2 1 2 1 1 2}$. Keep appending groups of digits each different from the previous ones by an additional digit 1. This guarantees that there are no groups of repeating decimals. The number is $\mathbf{0 . 2 1 2 1 1 2 1 1 1 2 1 1 1 1 2 1 1 1 1 1 2} \ldots$

### 4.3 Addition of Signed Numbers

One method is to think of the balance in your checking account.
$\mathbf{5}+\mathbf{2}=\mathbf{7}$. You start with $\$ \mathbf{5}$ in your checking account and deposit $\$ \mathbf{2}$. You now have $\$ \mathbf{7}$ in your account.
$\mathbf{5}+(\mathbf{- 2 )}=\mathbf{3}$. You start with $\$ 5$ in your checking account and write a check for $\$ \mathbf{2}$. You now have $\$ \mathbf{3}$ in your account. (The purpose of the parentheses is to separate + and - )
$-\mathbf{5}+\mathbf{2}=\mathbf{- 3}$. You start with $\$-\mathbf{5}$ in your checking account (you are overdrawn) and deposit $\$ \mathbf{2}$. You now have $\$-\mathbf{3}$ in your account (you are still overdrawn $\$ \mathbf{3}$ ).
$-5+(-2)=-7$. You start with $\$-5$ in your checking account (you are overdrawn) and write a check for $\$ \mathbf{2}$. You now have $\$-\mathbf{7}$ in your account (you are overdrawn $\$ \mathbf{7}$ ).

Another method is to use arrows.
A positive number is represented by an arrow pointing in the positive direction (to the right). The length of the arrow is the number of units - tick marks on a number line - indicated by the number (plus one).
The number 5 is represented by


A negative number is represented by an arrow pointing in the negative direction (to the left). The length of the arrow is the number of units - tick marks on a number line - indicated by the number (plus one).

The number $\mathbf{- 2}$ is represented by

$-5+2=-3$.

$-5+(-2)=-7$.


Still another method is to follow a set of rules.
To Add two numbers of the same sign:
$5+\mathbf{2}=\mathbf{7}$ Obtain the absolute value of each number.
$|5|=5$ and $|2|=2$.
Add the absolute values $\mathbf{5}+\mathbf{2}=\mathbf{7}$.
The sign of the sum is the sign of the original numbers, namely positive.
Another example:
$(-\mathbf{5})+(-\mathbf{2})=-\mathbf{7}$ Obtain the absolute value of each number. $|-5|=5$ and $|-2|=\mathbf{2}$.
Add the absolute values $\mathbf{5}+\mathbf{2}=\mathbf{7}$.
The sign of the sum is the sign of the original numbers, namely negative.

To Add two numbers of opposite sign:
$-\mathbf{5}+\mathbf{2}=\mathbf{- 3}$ Obtain the absolute value of each number.
$|-5|=5$ and $|2|=2$.
Subtract the smaller absolute values from the larger one
$5-2=3$.
The sign of the sum is the sign of the original number that led to the larger absolute value, namely negative. Another example:
$\mathbf{5}+(\mathbf{- 2 )}=\mathbf{3}$ Obtain the absolute value of each number.
$|5|=5$ and $|-2|=2$.
Subtract the smaller absolute values from the larger one
$5-2=3$.
The sign of the sum is the sign of the original number that led to the larger absolute value, namely positive.
Here is one further method involving mathematical manipulatives suggested by the University of Utah ( nlvm.usu.edu/en/nav/frames_asid_161_g_2_t_1.html?from=search.html)
$5+\mathbf{2} \Rightarrow(+++++)$ added to $(++)=+++++++=7$
$5+(-2) \Rightarrow(+++++)$ added to $(--)=(+-)(+-)+++=3$
$-5+2 \Rightarrow(-----)$ added to
$(++)=(+-)(+-)---=-3$
$-5+(-2) \Rightarrow(-----)$ added to
$(--)=------=-7$
Each +- eliminate each other resulting in $\mathbf{0}$.

### 4.4 Subtraction of Signed Numbers

Replace subtraction of signed numbers by addition of signed numbers by making two changes:

1) change minus to plus.
2) Add the opposite of the number you originally subtracted.
$5-2=5+(-2)=3$
$5-(-2)=5+(+2)=7$
$-5-2=-5+(-2)=-7$
$-5-(-2)=-5+(2)=-3$
Be careful about the meaning of "difference" and "subtract from".
The difference of $\mathbf{9}$ and 5 is $\mathbf{9 - 5}=\mathbf{4}$,
but
subtract 9 from 5 is $\mathbf{5 - 9}=-\mathbf{4}$.

### 4.5 Multiplication of Signed Numbers

Multiplication is repeated addition.
$4(-3)=(-3)+(-3)+(-3)+(-3)=-12$
We can use this reasoning whenever we multiply two integers of different signs. Thus the product of a positive number and a negative number is negative.

Rewrite $\mathbf{4 ( - 3 )}$ as $-[\mathbf{4 ( 3 )}]$, that is isolate the negative sign and carry out the multiplication of what is left from the product.

Isolate the negative sign.
$(-\underbrace{4)(-3)}=-[4(-3)]=-[-12]=\mathbf{1 2}$
$(-3)^{2}=(-3)(-3)=\mathbf{9} \quad$ The opposite of $-\mathbf{1 2}$.
but
$-3^{2}=-(3)(3)=-9$ Be careful. Lots of students use exponentiation of signed numbers incorrectly.
Those who claim that $-\mathbf{5}^{\mathbf{2}}=\mathbf{2 5}$ belong to the IPS (Invisible Parentheses Society). If you are a member of the IPS, give up your membership immediately.

Warning:
Do not say "negative AND negative is positive." AND s usually taken to mean addition. The sum of two negative numbers is negative.

Say instead "The product (or quotient for division) of two negative numbers is positive."

### 4.6 Division of Signed Numbers

Division is repeated subtraction.

$$
\frac { 1 2 } { 4 } = 1 2 \div 4 \Rightarrow 4 \longdiv { 1 2 }
$$

$$
-4 \bigcirc
$$

$$
8
$$

$$
-4
$$

$$
4
$$



We can subtract 4 from 12
exactly 3 times.
0
-4 Cannot be negative
Here is a more useful definition of division in terms of multiplication.
$12 \div 4=\frac{12}{(4)}=3$ then $\underbrace{4 \times 3}=12$
$12 \div(-3)=\frac{12}{-3}=-4$ because $(-3)(-4)=12$
$-12 \div(3)=\frac{-12}{3}=-4$ because $(3)(-4)=-12$
$-12 \div(-3)=\frac{-12}{-3}=4$ because $(-3)(4)=-12$
The sign rules of a quotient are identical to the product rules:
The quotient of two same signed numbers is positive.
The quotient of two opposite signed numbers is negative.
Division involving 0 :
$0 \div(3)=\frac{0}{3}=0$ because $(0)(3)=0$
$12 \div(0)=\frac{12}{0}=x$.
$\boldsymbol{x}$ is undefined because $(\mathbf{0})(\boldsymbol{x})=\mathbf{0} \neq \mathbf{1 2}$.
The operation of division by 0 is undefined. It is incorrect to say that 0 is undefined (unless you need to explain your grades to your parents or spouse.)

Example 1:
Evaluate $\frac{2 x-4(y+3)}{3 x-y+7}$ if $x=-3$ and $y=-2$.
Solution;

$$
\begin{aligned}
\frac{2 x-4(y+3)}{3 x-y+7} & =\frac{2()-4(()+3)}{3()-()-7} \quad \text { Replace variables by parentheses. } \\
& =\frac{2(-3)-4((-2)+3)}{3(-3)-(-2)+7} \quad \text { Drop values of variables into }() \\
& =\frac{-6-4(1)}{-9-(-2)+7} \\
& =\frac{-6-4}{-9+(2)+7} \\
& =\frac{-10}{-7+7} \\
& =\frac{-10}{0} \text { which is undefined. }
\end{aligned}
$$

## Exercise 4

1. Name a whole number that is not a counting number.
2. Why is every integer a rational number?
3. Which rational number does not have a reciprocal?
4. There are infinitely many irrational numbers between any two rational numbers. Construct an irrational number between $\frac{1}{5}$ and $\frac{1}{6}$.
5. 

| $12+(-7)=$ | $-11+9=$ | $-14+(-7)=$ | $10-(-7)=$ |
| ---: | :---: | :---: | :---: |
| $-16-(-25)=$ | $-11-19=$ | $-8-(-5)=$ | $20-(-7)=$ |

6. 

| $14(-2)=$ | $-11 \cdot 9=$ | $-14 \cdot(-7)=$ | $70 \div(-7)=$ |
| ---: | ---: | ---: | ---: |
| $-100 \div[-(-25)]=$ | $-19 \div(-38)=$ | $-[-8 \div(-40)]=$ | $20 \div(-85)=$ |

7. What number is subtracted from $\mathbf{- 1 0}$ to get a difference of $\mathbf{3 0}$ ?
8. Mount McKinley in Alaska is at $\mathbf{2 0}, \mathbf{3 0 0} \mathrm{ft}$ above sea level and Death Valley in California is at $\mathbf{2 8 0} \mathrm{ft}$ below sea level. Draw a vertical number line to identify altitude, with positive upward. Place a dot at sea level, at Death Valley and at Mount McKinley. Label each point with a signed number. What is the distance (in ft) between Mount McKinley and Death Valley?
9. The temperature on a cold Winter day in Luxembourg was measured four times. Three measurements were $\mathbf{4}^{\circ} \mathrm{C},-\mathbf{7}^{\circ} \mathrm{C}$, and $\mathbf{8}^{\circ} \mathrm{C}$. The average was $-\mathbf{1}^{\circ} \mathrm{C}$. The fourth measurement was lost. Find the lost measurement if the average for that day was $-1^{\circ} \mathrm{C}$
10. A decorator from Exquisite Affairs blows helium balloons for a birthday celebration. Each balloon is tied to a 9 ft string. The decorator has 256 yds of string on a roll. Does the decorator have enough string to blow 100 balloons? If yes, is there any string left? If no, how many balloons can be blown? There are 3 feet per yard.
11. A restaurant cook has 25 kilograms ( kg ) of potatoes. Each order on a banquet menu requires 3 potatoes on the average. Assume the average weight of a potato to be 100 grams (g). There are 27 people attending the banquet. Does the cook have enough potatoes to serve the banquet? There are $1,000 \mathrm{~g}$ in a kg .

## STOP!

The solutions follow:
Keep them covered up till you have worked out each of the problems above.

1. Name a whole number that is not a counting number.

## Solution:

The set (collection) of whole numbers is $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots\}$ All of these numbers are alo counting numbers except for $\mathbf{0}$. Thus $\mathbf{0}$ is a whole number (the only whole number) that is not a counting number.
2. Why is every integer a rational number?

## Solution:

Every integer, like 6 , can be written as a rational number (ratio, fraction) whose denominator is 1 . $6=\frac{6}{1}$.
3. Which rational number does not have a reciprocal?

## Solution:

$\mathbf{0}$ does not have a reciprocal because $\frac{\mathbf{1}}{\mathbf{0}}$ does not exist.
4. There are infinitely many irrational numbers between any two rational numbers. Construct an irrational number between $\frac{\mathbf{1}}{5}$ and $\frac{\mathbf{1}}{6}$.

## Solution:

$$
\begin{aligned}
& \frac{1}{5}=0.2 \\
& \frac{1}{6}=0.1666 \cdots
\end{aligned}
$$

The number $\boldsymbol{N}$ to be constructed must not have any group of repeating decimals, as follows (the answer is not unique):
$N=0.17117111711117 \ldots$

## 5. Solution:

$$
\begin{aligned}
& 12+(-7)=5, \|-11+9=-2, \\
& -14+(-7)=-21,10-(-7)=17, \\
& -16-(-25)=9, \quad-11-19=-30, \\
& -8-(-5)=-3,20-(-7)=27,
\end{aligned}
$$

## 6. Solution:

$$
\left.\begin{array}{rlc}
14(-2) & = & \left.-28, \left\lvert\, \begin{array}{rlll}
-11 \cdot 9 & = & -99 \\
-14 \cdot(-7) & = & 98, & 70 \div(-7)
\end{array}\right.\right) \\
-100 \div[-(-25)] & = & -4, \\
-19 \div(-38) & = & \frac{1}{2} \\
-[-8 \div(-40)] & = & -\frac{1}{5},
\end{array} \right\rvert\, \begin{array}{rll}
17
\end{array}
$$

7. What number is subtracted from $\mathbf{- 1 0}$ for a difference of $\mathbf{3 0}$ ?

## Solution:

Let $\boldsymbol{x}$ be the number. Then $-\mathbf{1 0}-\boldsymbol{x}=\mathbf{3 0} . \boldsymbol{x}$ has to be negative because subtracting a negative number is equivalent to adding a positive number. Then $x=-40$ because $-10-(-40)=30 \Rightarrow-10+40=30$.
8. Mount McKinley in Alaska is at 20, $\mathbf{3 0 0} \mathrm{ft}$ above sea level and Death Valley in California is at $\mathbf{2 8 0} \mathrm{ft}$ below sea level. Draw a vertical number line to identify altitude, with positive upward. Place a dot at sea level, at Death Valley and at Mount McKinley. Label each point with a signed number. What is the distance (in ft) between Mount McKinley and Death Valley?

## Solution:


9. The temperature on a cold Winter day in Luxembourg was measured four times. Three measurements were $4^{\circ} \mathrm{C},-7^{\circ} \mathrm{C}$, and $\mathbf{8}^{\circ} \mathrm{C}$. The average was $-\mathbf{1}^{\circ} \mathrm{C}$. The fourth measurement was lost. Find the lost measurement if the average for that day was $-\mathbf{1}^{\circ} \mathrm{C}$

## Solution:

Since the average of the four temperatures is $\mathbf{- 1}$, the sum of the four temperatures is $(4)(-1)=-4$. The sum of the three recorded temperatures is $(4)+(-7)+(8)=-3+8=5$. The lost temperature is $\mathbf{- 4 - 5}=-\mathbf{9}^{\circ} \mathrm{C}$.
10. A decorator from Exquisite Affairs blows helium balloons for a birthday celebration. Each balloon is tied to a 9 ft string. The decorator has 256 yds of string on a roll. Does the decorator have enough string to blow 100 balloons? If yes, is there any string left? If no, how many balloons can be blown? There are 3 feet per yard.

## Solution:

$\frac{256 \mathrm{yds}}{1} \cdot \frac{3 \mathrm{ft}}{\mathrm{yd}}=\mathbf{7 6 8} \mathrm{ft}$.
The first balloon requires 9 ft of string. The decorator has $\mathbf{7 6 8} \mathbf{- 9}=\mathbf{7 5 9} \mathrm{ft}$ left.
The second balloon requires 9 ft of string. The decorator has $\mathbf{7 5 9} \mathbf{- 9}=\mathbf{7 5 0} \mathrm{ft}$ left.
This repeated subtraction of 9 ft implies division.
The number of balloons that can be blown is
$\frac{768 \mathrm{ft}}{1} \cdot \frac{1 \text { balloon }}{9 \mathrm{ft}}=85 \frac{1}{3}$ balloons.
The decorator can blow (and tie) $\mathbf{8 5}$ balloons. She needs to tie $\mathbf{1 0 0} \mathbf{- 8 5}=\mathbf{1 5}$ more balloons. She is short $\mathbf{1 5} \cdot \mathbf{9} \mathrm{ft}=\mathbf{1 3 5}$ more ft of string. (The 3 ft of sting left from the roll of string is wasted.)
11. A restaurant cook has 25 kilograms ( kg ) of potatoes. Each order on a banquet menu requires 3 potatoes on the average. Assume the average weight of a potato to be 100 grams (g). There are 27 people attending the banquet. Does the cook have enough potatoes to serve the banquet? There are $1,000 \mathrm{~g}$ in a kg .

## Solution:

The cook needs to fill 27 orders.
The number of potatoes needed is
$\frac{\mathbf{2 7} \text { orders }}{\mathbf{1}} \cdot \frac{\mathbf{3} \text { potatoes }}{\mathbf{1} \text { order }}=\mathbf{8 1}$ potatoes.
The 81 potatoes weigh $\frac{\mathbf{8 1} \text { potatoes }}{\mathbf{1}} \cdot \frac{\mathbf{1 0 0} \mathrm{g}}{\mathbf{1} \text { potato }}=\mathbf{8}, \mathbf{1 0 0} \mathrm{g}$.
The number of kg needed is $\frac{8,100 \mathrm{~g}}{1} \cdot \frac{1 \mathrm{~kg}}{1,000 \mathrm{~g}}=\mathbf{8 . 1} \mathrm{kg}$.
The cook has $\mathbf{2 5} \mathrm{kg}$ of potatoes, which is about $\mathbf{3}$ times what is needed.

## Chapter 5

## Fraction Notation and Percent

### 5.1 Youtube

http://www.youtube.com/playlist?list=PLC1ACD73D1CE0A2F8\&feature=view_all

### 5.2 Fraction Concept

A fraction is a portion of a whole. Half a pie, two-thirds of a cake, a quarter of an hour. A whole can be any quantity or object. A whole pencil, a whole car, a whole inch.

We shall use a basic rectangle (like a sheet of paper $\mathbf{8} \frac{\mathbf{1}}{\mathbf{2}} \times \mathbf{1 1}$ inches) as a whole.



Num. $\rightarrow 1$
Den. $\frac{\mathbf{1}}{\mathbf{3}}$ means 1 part out of 3


A proper fraction is a fraction whose numerator is less than its denominator.


$$
\frac{\mathbf{2 0}}{\mathbf{1 5}}=\frac{\mathbf{4}}{\mathbf{3}}=\mathbf{1} \frac{\mathbf{1}}{\mathbf{3}} \quad \begin{aligned}
& \begin{array}{l}
\text { A fraction whose numerator is equal to or more than its } \\
\text { denominator is an improper fraction. }
\end{array} \\
& \\
& \text { An improper fraction can be turned into a mixed number. }
\end{aligned}
$$

$\frac{\mathbf{3 5}}{\mathbf{6}}=\mathbf{5}+\frac{\mathbf{5}}{\mathbf{6}}=\mathbf{5} \frac{\mathbf{5}}{\mathbf{6}}$ (simply divide 35 by 6.$)$
There is an implied addition (plus) sign between the whole number and the proper fraction.
A mixed number can be turned into an improper fraction, like
$7 \frac{2}{5}=\frac{7}{1}+\frac{2}{5}=\frac{7 \cdot 5}{5}+\frac{2}{5}=\frac{35}{5}+\frac{2}{5}=\frac{37}{5}$
Shortcut: $7 \frac{2}{5}=\frac{7 \cdot 5+2}{5}$

### 5.3 Equivalent Fractions



Since 1 is the identity for multiplication, any number (including a fraction) times 1 is that number.
$\frac{1}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \cdot \frac{2}{2}=\frac{2}{4}$
$\frac{1}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \cdot \frac{3}{3}=\frac{3}{6}$
$\frac{1}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \cdot \frac{4}{4}=\frac{4}{8}$
$\frac{1}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \cdot \frac{5}{5}=\frac{5}{10}$
The examples above illustrate that
The numerator and and the denominator of any fraction can be multiplied by the same number
to get similar fractions. The key word here is multiplied.
The process works in reverse. When we can factor out a $\mathbf{1}$, we obtain a reduced fraction. This leads to a simplified fraction.

$$
\begin{aligned}
& \frac{2}{4}=\frac{1}{2} \cdot \frac{2}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \\
& \frac{3}{6}=\frac{1}{2} \cdot \frac{3}{3}=\frac{1}{2} \cdot 1=\frac{1}{2} \\
& \frac{4}{8}=\frac{1}{2} \cdot \frac{4}{4}=\frac{1}{2} \cdot 1=\frac{1}{2}
\end{aligned}
$$

$\frac{5}{10}=\frac{1}{2} \cdot \frac{5}{5}=\frac{1}{2} \cdot 1=\frac{1}{2}$
The examples above illustrate that

The numerator and and the denominator of any fraction can be divided by the same number
to get simplified fractions. This means we factor out a 1. The key word here is factored. Factoring means that the numerator and denominator must be factorable.


Be very careful with the word "cancel".

$$
1=\frac{3+6}{9}=\frac{3(1+2)}{(3)(3)}=\frac{1+2}{3} \cdot \frac{3}{3}=\frac{1+2}{3} \cdot 1=\frac{3}{3} \cdot 1=1
$$

If possible, always reduce fractions given as answers.

The divisibility tests come in handy.
A number is divisible by $\mathbf{2}$ if its last (ones place) digit is $\mathbf{0}, \mathbf{2}, \mathbf{4}, \mathbf{6}$, or $\mathbf{8}$. Such a number is even number.

$$
\frac{1234}{2}=617
$$

A number is divisible by $\mathbf{3}$ if the sum of its digits is divisible by $\mathbf{3}$.
$\mathbf{1 , 2 4 9}$ is not divisible by 3 because $\mathbf{1 + 2 + 4 + 9 = 1 6}$ which canot be divided by $\mathbf{3}$.

$$
\frac{12147}{3}=4,049
$$

because $1+2+1+4+\mathbf{7}=\mathbf{1 5}$ which can be divided by 3 .
A number is divisible by 4 if the number represented by the the last two digits is divisible by 4.
$\frac{1236}{4}=309$
because $\mathbf{3 6}$ in $\mathbf{1 2 \boxed { 3 6 }}$ is divisible by $\mathbf{4}$.
A number is divisible by $\mathbf{5}$ if its last (ones place) digit is $\mathbf{0}$ or $\mathbf{5}$.
$\frac{1235}{5}=247$
A number is divisible by 6 if it is divisible by 2 AND $\mathbf{3}$ simultaneously.
$\frac{1230}{6}=205$
because $\mathbf{1 , 2 3 0}$ is even and $\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{0}=\mathbf{6}$.
For divisibility by $\mathbf{7}$, just perform a long division and check if there is a remainder. You can find methods on the internet, but the procedures take as long or longer than long division. Make sure you remember your multiplication table.

A number is divisible by 8 if the number represented by the the last three digits is divisible by 8 .

$$
\frac{1232}{8}=154
$$

because 232 in $\mathbf{1} 232$ is divisible by 8.
A number is divisible by 9 if the sum of its digits is divisible by 9.
1,269 is divisible by 9 because $1+2+6+9=18$ which can be divided by 9 .
$\frac{1269}{9}=141$
because $\mathbf{1}+\mathbf{2 + 6 + 9 = 1 8}$ which can be divided by $\mathbf{9}$.
A number is divisible by $\mathbf{1 0}$ if its last (ones place) digit is $\mathbf{0}$.
$\frac{1,230}{10}=123$
A number is divisible by 11 if the difference of the sums of its even-positioned and odd-positioned digits is a multiple of $\mathbf{1 1}$ or $\mathbf{0}$.

$$
\begin{aligned}
& \frac{27,269}{11}=2,479 \\
& \text { 27,269 } \\
& 2+2+9=13
\end{aligned}
$$

Example 1:
Given $\frac{\mathbf{3}}{\mathbf{7}}$ find the equivalent fraction whose denominator is $\mathbf{2 1} \boldsymbol{x}^{\mathbf{2}}$.

## Solution:

$\left.21 x^{2}=7 \dot{(3} x^{2}\right)$
Then $\frac{3}{7}=\frac{3}{7} \cdot \frac{3 x^{2}}{3 x^{2}}=\frac{9 x^{2}}{21 x^{2}}$

Example 2:
Reduce $\left(\frac{2}{35}\right)\left(\frac{18}{11}\right)\left(\frac{12}{49}\right)\left(\frac{455}{432}\right)$

## Solution:

$$
\begin{aligned}
& \left(\frac{2}{35}\right)\left(\frac{18}{11}\right)\left(\frac{12}{49}\right)\left(\frac{455}{432}\right) \\
= & \left(\frac{2}{5 \cdot 7}\right)\left(\frac{18}{11}\right)\left(\frac{12}{49}\right)\left(\frac{5 \cdot 91}{432}\right) \\
=\left(\frac{2}{7}\right)\left(\frac{18}{11}\right)\left(\frac{4 \cdot 3}{49}\right)\left(\frac{91}{4 \cdot 108}\right) & \text { Factor } \frac{5}{5} \\
=\left(\frac{2}{7}\right)\left(\frac{9 \cdot 2}{11}\right)\left(\frac{3}{49}\right)\left(\frac{91}{9 \cdot 12}\right) & \text { Factor } \frac{4}{4} \frac{9}{9} \\
=\left(\frac{2}{7 \cdot 1}\right)\left(\frac{2}{11}\right)\left(\frac{3}{49}\right)\left(\frac{7 \cdot 13}{12}\right) & \text { Factor } \frac{7}{7} \\
=\left(\frac{2}{1}\right)\left(\frac{2}{11}\right)\left(\frac{3}{49}\right)\left(\frac{13}{2 \cdot 2 \cdot 3}\right) & \text { Factor } \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3} \\
= & \left(\frac{1}{1}\right)\left(\frac{1}{11}\right)\left(\frac{1}{49}\right)\left(\frac{13}{1 \cdot 1 \cdot 1}\right) \\
= & \frac{13}{11 \cdot 49}=\frac{13}{539}
\end{aligned}
$$

### 5.4 Addition of Fractions



$$
\frac{1}{2}=\frac{3}{6}
$$


$\frac{1}{3}=\frac{2}{6}$


4 in


We need to add $\frac{\mathbf{1}}{\mathbf{2}}$ to $\frac{\mathbf{1}}{\mathbf{3}}$.
$\frac{\mathbf{1}}{\mathbf{2}}$ of the whole (labeled APPLE) is a rectangle with dimensions 12 by 4 inches.
$\frac{\mathbf{1}}{\mathbf{3}}$ of the whole (labeled ORANGE) is a rectangle with dimensions 8 by 2 inches.
We cannot add apples to bananas (unless you make a cocktail salad).

We can turn the APPLE into 3 BANANAs (dimensions 4 by 4 inches).
We can turn the ORANGE into 2 BANANAs.
Adding an APPLE to an ORANGES is equivalent to adding 3 BANANAs to 2 BANANAs.
Mathematically, we obtained a common denominator and then added the numerators.

$$
\begin{aligned}
\frac{1}{2} & =\frac{1}{2} \cdot \frac{3}{3}=\frac{3}{6} \\
\frac{1}{2} & =\frac{1}{3} \cdot \frac{2}{2}=\frac{2}{6} \\
& =\quad=\frac{3+2}{6}=\frac{5}{6}
\end{aligned}
$$

If our reasoning is correct, then the area of the APPLE and the ORANGE should equal the area of 5 BANANAs.
$4 \cdot 12+8 \cdot 4=48+32=80$ in $^{2}($ APPLE + ORANGE $)$
$\mathbf{5} \cdot(\mathbf{4} \times \mathbf{4})=\mathbf{5} \cdot \mathbf{1 6}=\mathbf{8 0} \mathrm{in}^{2}(5$ BANANAs $)$
The original whole has $12 \times 8=108 \mathrm{in}^{2}$.
$\frac{5}{6} \cdot 108=\frac{5}{6} \cdot \frac{108}{1}=\frac{5 \cdot 12 \cdot 8}{6}=\frac{5 \cdot 2 \cdot 8}{1}=(5 \cdot 2)(8)=10 \cdot 8=80$
$\mathrm{in}^{2}$ which is another geometrical check.

### 5.5 Least Common Multiple (Denominator)

Let's add $\frac{2}{15}+\frac{\mathbf{3}}{\mathbf{1 0}}$.
The procedure in the previous section leads to

$$
\begin{aligned}
& \frac{2}{15}=\frac{2}{15} \cdot \frac{10}{10}=\frac{20}{150} \\
& \frac{3}{10}=\frac{3}{10} \cdot \frac{15}{15}=\frac{45}{150} \\
& =\frac{20+45}{150}=\frac{65}{150}=\frac{5 \cdot 13}{3 \cdot 5 \cdot 10}=\frac{13}{30} \text { Factor } \frac{5}{5}=1
\end{aligned}
$$

The denominator $\mathbf{1 5 0}$ in this problem is larger than needs to be. The largest denominator we need is the Least Common Multiple of the denominators (called Least Common Denominator - LCD) is obtained after writing the prime factorization of each denominator. The divisibility tests are handy here.


Example 3: Find the LCD of the denominators 264 and 756.


$$
\begin{array}{ll|l|l|l|l|l|l|l|l|}
264 & = & 2 & 2 & 2 & 3 & & & & 11 \\
756 & = & 2 & 2 & & 3 & 3 & 3 & 7 & \\
\hline \text { LCD } & = & 2 & 2 & 2 & 3 & 3 & 3 & 7 & 11
\end{array}=756(2)(11)=16,632
$$

Just as a quick check, $\frac{16,632}{756}=22$ and $\frac{16,632}{264}=63$
Were you taught to find the LCM by using sets of multiples?
Find the LCM of $\mathbf{2}$ and $\mathbf{3}$ :
Multiples of 2 : $\{2,4,6,8,10,12, \cdots\}$
Multiples of $\mathbf{3}:\{\mathbf{3}, \mathbf{6}, \mathbf{9}, \mathbf{1 2}, \cdots\}$
$\mathbf{6}, \mathbf{1 2}, 18, \cdots$ are multiples of $\mathbf{2}$ and $\mathbf{3 .} \mathbf{6}$ is the smallest (Least) multiple Common to both $\mathbf{2}$ and $\mathbf{3}$.
Do you see the difficulty in using this procedure to find the LCM of $\mathbf{2 6 4}$ and $\mathbf{7 5 6}$ ?
Multiples of 264: $\{\mathbf{2 6 4}, \mathbf{5 2 8}, \mathbf{7 9 2}, \cdots\}$
Multiples of 756: $\{\mathbf{7 5 6}, \mathbf{1 5 1 2}, \mathbf{2 2 6 8}, \cdots\}$

### 5.6 Addition of Fractions

Example 4:
Add $\frac{7}{264}+\frac{5}{756}$

## Solution:

A common denominator is needed when adding fractions. We found that the LCD of $\mathbf{2 6 4}$ and $\mathbf{7 5 6}$ is 16, 632.

Note from the LCD array above that 264 needs to be multiplied by $(3)(3)(7)=(9)(7)=63$ to get 16, 632.

Similarly 756 needs to be multiplied by $(2)(11)=22$ to get $\mathbf{1 6}, 632$.

$$
0.0265152+0.0066138=0.033129(\text { Nice! })
$$

### 5.7 Subtraction of Fractions

Rewrite subtraction as addition of opposites.

### 5.8 Multiplication of Fractions

What is two-thirds of a quarter?


The original whole
The new whole
The set of pictures is intended to show that $\frac{\mathbf{1}}{\mathbf{3}}$ of $\frac{\mathbf{1}}{\mathbf{4}}$ is $\frac{\mathbf{1}}{\mathbf{1 2}}$
that is

$$
\frac{1}{3} \cdot \frac{1}{4}=\frac{1}{12}
$$

$$
\begin{aligned}
& \frac{7}{264}=\frac{7}{264} \frac{63}{63} \quad=\frac{441}{16,632} \quad \text { Quick check: }=0.0265152 \\
& +\frac{5}{756}=\frac{5}{756} \frac{22}{22}=\frac{110}{16,632} \quad \text { Quick check: }=0.0066138 \\
& =\frac{441+110}{16,632}=\frac{551}{16,632} \text { Quick check: }=0.03312890
\end{aligned}
$$

The last statement suggests that we can multiply numerators and denominators of the product of fractions. Reduce fractions before multiplying first, if possible.

### 5.9 Division of Fractions

Divide $\frac{\frac{\mathbf{2}}{\mathbf{3}}}{\frac{\mathbf{5}}{\mathbf{7}}}$
Multiply the fraction by $\frac{\frac{\mathbf{7}}{\mathbf{5}}}{\frac{\mathbf{7}}{\mathbf{5}}}=1$
to get

$$
\frac{\frac{2}{3}}{\frac{5}{7}} \cdot \frac{\frac{7}{5}}{\frac{7}{5}}=\frac{\frac{2}{3} \cdot \frac{7}{5}}{\frac{1}{1}}=\frac{2}{3} \cdot \frac{7}{5}
$$

This example illustrates that division by a fraction is equivalent to multiplication by the reciprocal of the original divisor (denominator).

$$
\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b} \cdot \frac{d}{c}
$$

### 5.10 Mixed Numbers

A mixed number is a "mixed up" number. It cannot decide whether it is an improper fraction or a whole number. It is a whole number added to a proper positive fraction, with the addition sign omitted.


The numerator of a proper fraction, like $\frac{\mathbf{1 1}}{\mathbf{1 2}}$, is less than its denominator.


The numerator of an improper fraction, like
$1 \frac{5}{12}$, is equal to or more
than its denominator.

Here is how to convert an improper fraction to a mixed number. Divide the numerator of an improper fraction by its denominator. The quotient is the whole number. The remainder is the numerator of the proper fraction.

$$
\frac{41}{12}=3 \frac{5}{12}
$$

Now convert $5 \frac{\mathbf{4}}{\mathbf{7}}$ to an improper fraction.

$$
\begin{aligned}
5 \frac{4}{7} & =5 \frac{4}{7} \\
& =\frac{5}{1}+\frac{4}{7} \\
& =\frac{5 \cdot 7}{7}+\frac{4}{7} \\
& =\frac{35}{7}+\frac{4}{7} \\
& =\frac{35+4}{7} \\
& =\frac{39}{7}
\end{aligned}
$$

Shortcut: $5 \frac{4}{7}=\frac{5 \cdot 7+4}{7}=\frac{39}{7}$

### 5.11 Arithmetic of Mixed Numbers

### 5.11.1 Addition of Mixed Numbers

Example 5:
Add $4 \frac{3}{5}+2 \frac{1}{3}$

## Solution:

$$
\begin{aligned}
\left.4 \frac{3}{5}=\frac{4(5)+3}{5}=\frac{23}{5} \right\rvert\,=\frac{23}{5} \cdot \frac{3}{3} & =\frac{115}{15} \\
\left.+2 \frac{1}{3}=\frac{2(3)+1}{3}=\frac{7}{3} \right\rvert\,=\frac{7}{3} \cdot \frac{5}{5} & =\frac{35}{15} \\
& =\frac{115+35}{15}=\frac{150}{15}=10
\end{aligned}
$$

Example 6:
Add $87,654,321 \frac{4}{5}+12,345,678 \frac{2}{3}$

## Solution:

$$
\begin{aligned}
& 87,654,321 \frac{4}{5}=\left\lvert\, \frac{4}{5}=\frac{4 \cdot 3}{5 \cdot 3}=\frac{12}{15}\right. \\
& +12,345,678 \frac{2}{3}=\frac{2}{3}=\frac{2}{3} \cdot \frac{5}{5}=\frac{10}{15} \\
& \left\lvert\,=\frac{27}{15}=1+\frac{12}{15}=1+\frac{4}{5}\right.
\end{aligned}
$$

The method used in example 5 can be used, but it leads to very large numbers. Instead we add the whole numbers and the proper fractions individually.

The sum is $87,654,321+12,345,678+1+\frac{4}{5}=100,000,000 \frac{4}{5}$.

### 5.11.2 Subtraction of Mixed Numbers

Example 7:
Subtract $59 \frac{1}{6}-3 \frac{4}{5}$

## Solution:

The method of example 5 is straightforward, but suppose the numbers are large. Then the following technique is handy.

$$
\begin{aligned}
59 \frac{1}{6} & =\left\lvert\, \begin{array}{r}
\frac{1}{6} \\
-\quad 3 \frac{4}{5}
\end{array}=\frac{1 \cdot 5}{6 \cdot 5}\right.
\end{aligned}=\frac{5}{30} \frac{-4}{5}=\frac{-4}{5} \cdot \frac{6}{6}=\frac{-24}{30}, ~=\frac{-19}{30}
$$

The difference is $59-3+\frac{-19}{30}=56+\frac{-19}{30}$. If the proper
fraction were positive, we would be finished. But here we shall rewrite $\mathbf{5 6}$ as $\mathbf{5 5}+\mathbf{1}$ and convert the 1 to an equivalent fraction with denominator 30 . This is called borrowing.

$$
\begin{aligned}
& 59-3+\frac{-19}{30}=56+\frac{-19}{30} \\
&=55+1+\frac{-19}{30} \\
&=55+\frac{30}{30}+\frac{-19}{30} \\
&=55+\frac{30-19}{30} \\
&=55+\frac{11}{30} \\
& 59 \frac{1}{6}-3 \frac{4}{5}=55 \frac{11}{30}
\end{aligned}
$$

### 5.11.3 Multiplication of Mixed Numbers

Convert the mixed numbers to improper fractions, then multiply the fractions. Do not forget about reducing fractions. When finished with the product, convert that product to a mixed number.

### 5.11.4 Division of Mixed Numbers

Convert the mixed numbers to improper fractions, then divide the fractions (multiply by the reciprocal of the divisor (denominator)). Do not forget about reducing fractions. When finished with the product (which was originally a quotient), convert that product to a mixed number.

### 5.12 Complex Fractions

A complex fraction is a fraction whose numerator and/or denominator is a fraction.
Example 5:
Simplify $\frac{\frac{1}{2}-\frac{3}{5}}{\frac{1}{4}-\frac{9}{25}}$

## Solution:

One method is to combine the fractions in the numerator and/or denominator into single fractions. Then multiply the numerator by the reciprocal of the denominator.

Another method is to multiply both numerator and denominator of all the fractions by the LCD of all the fractions.

The LCD of $\mathbf{2}, \mathbf{5}, \mathbf{4}$, and $\mathbf{2 5}$ is $\mathbf{1 0 0}$.

$$
\begin{aligned}
& \frac{\frac{1}{2}-\frac{3}{5}}{\frac{1}{4}-\frac{9}{25}} \cdot \frac{100}{100}=\frac{\frac{1(100)}{2}-\frac{3(100)}{5}}{\frac{1(100)}{4}-\frac{9(100)}{25}}=\frac{1(50)-3(20)}{1(25)-9(4)} \\
= & \frac{50-60}{25-36}=\frac{10}{11}
\end{aligned}
$$

Do not confuse complex fractions with complex numbers. You will learn about complex numbers toward the end of this course.

### 5.13 Convert a Decimal Number with Repeating Decimals to a Rational Number

We illustrate the technique on an example.
Example 6:
Convert $N=0.3575757 \cdots 57 \cdots=0.3 \overline{57}$ to a rational number (a ratio of two integers).

## Solution:

The trick is to rewrite $\boldsymbol{N}$ such that the decimal point occurs before the first group of repeating decimals. Call this number $\boldsymbol{N}_{\mathbf{2}}$.

Then rewrite $\boldsymbol{N}$ as another number $\boldsymbol{N}_{\mathbf{1}}$ such that the decimal point occurs immediately after the first group of repeating decimals.

Subtract the $\boldsymbol{N}_{\mathbf{1}}-\boldsymbol{N}_{\mathbf{2}}$. The decimal points line up and the difference $\boldsymbol{N}_{\mathbf{1}}-\boldsymbol{N}_{\mathbf{2}}$ will contain infinitely many zeros after the decimal point. One dision (and possible reduction) will leave you with a rational number.

$$
\begin{aligned}
N & =0.3575757 \cdots 57 \cdots=0.3 \overline{57} \\
1,000 N=N_{1} & =357.5757 \cdots 57 \cdots
\end{aligned}
$$

$$
\begin{aligned}
1,000 N-10 N & =354.0000 \cdots 00 \cdots \text { Subtract } N_{2} \text { from } N_{1} \\
990 N & =354 \text { Subtract } N_{2} \text { from } N_{1} \\
N & =\frac{354}{990} \text { Divide both sides by } 990 \\
N=\frac{354}{990}=\frac{177}{495} & =\frac{59}{165}
\end{aligned}
$$

Check:

$$
\frac{354}{990}=0.35757575757575757575757575757576
$$

The rightmost 6 is due to rounding.

### 5.14 Percent

"of" frequently means multiplication in mathematics; "per" similarly means division.
Cent is found in words like century, cents, percent. It identifies a group of 100 .
"Percent" $\Rightarrow$ Per Cent refers to division by 100 .
Thus 5 percent means $\frac{\mathbf{5}}{\mathbf{1 0 0}}$ written $5 \%$.
Is percent useful?
You invest $\$ 655$ in a bank and earn $\$ 21.29$ in simple interest at the end of the year.
Your friend invests $\$ 967$ on the same day in another bank and earns $\$ 28.53$ at the end of the year.
Is your friend getting a better deal because his return is higher than yours?
It is impossible to decide without further computations. Suppose you and your friend invest the same amount, say $\$ 100$ at the same time in these banks. Suppose you earn $\$ 3.25$ while your friend earns $\$ 2.95$. Your bank has the better yield. Your bank pays $3.25 \%$ while your friend's bank returns $2.95 \%$.

Basing every comparison on 100 is quite useful, for the sake of consistency.
The following formula is extremely useful in dealing with percent

Amount is Percent (decimal) of Base

$$
A=P_{d} \cdot B
$$

$2.7=\frac{2.7}{1}=\frac{270}{100}=270 \%$
$2.7 \%=\frac{2.7}{100}=\frac{0.027}{1}=0.027$

Example 4:
What is $20 \%$ of 80 ?

## Solution:

$20 \%=0.2$

$$
A=P_{d} \cdot B
$$



What is $20 \%$ of 80 ?

$$
\begin{aligned}
& \stackrel{\downarrow}{\downarrow} \\
& \stackrel{\downarrow}{=} \\
& (0.2) \cdot(80) \\
& 16
\end{aligned}
$$

16 is $20 \%$ of 80 .

Example 5:
80 is $20 \%$ of what number?

## Solution:

$20 \%=0.2$


80 is $20 \%$ of what number?

$$
\begin{aligned}
80 & =(0.2) \cdot B \\
\frac{80}{0.2} & =\left(\frac{0.2}{0.2}\right) \cdot B \\
\frac{800}{2} & =B \\
B & =400
\end{aligned}
$$

80 is $\mathbf{2 0 \%}$ of $\mathbf{4 0 0}$.
Example 6:
20 is what percent of $80 ?$

## Solution:



20 is $25 \%$ of 80 .
You can also use $\frac{\boldsymbol{P}}{\mathbf{1 0 0}}=\frac{\boldsymbol{A}}{\boldsymbol{B}}$. The danger is that you interchange
the numbers for $\boldsymbol{A}$ and $\boldsymbol{B}$. The danger of error is lessened if you remember that $\boldsymbol{B}$ is the number following the word "of".

### 5.15 Exercise 5

1. Find $\mathbf{3}$ fractions equivalent to $\frac{8}{11}$ with numerators $\boldsymbol{n} \leq \mathbf{3 5}$.
2. Reduce $\frac{\mathbf{6 3 0}}{\mathbf{1 , 7 6 4}}$
3. Find the units (ones) digit $\boldsymbol{d}$ such that $\mathbf{4 9}, \mathbf{1 7}$ " $\boldsymbol{d}$ " is divisible by $\mathbf{2}$.
4. Find the units (ones) digit $\boldsymbol{d}$ such that $\mathbf{4 9}, \mathbf{1 7}$ " $\boldsymbol{d}$ " is divisible by $\mathbf{3}$.
5. If possible, construct a number that is divisible by all of the following numbers: $\mathbf{2 , 3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}$, and 10.
6. Reduce Reduce $\left(\frac{\mathbf{1 4}}{\mathbf{3 5}}\right)\left(\frac{\mathbf{3 6}}{\mathbf{7 7}}\right)\left(\frac{\mathbf{4 9}}{\mathbf{2 4}}\right)$
7. What is the prime factorization of $\mathbf{2 6 4}$ ?
8. What is the prime factorization of 495 ?
9. Find the Least Common Multiple of 264 and 495.
10. Add $\frac{5}{264}+\frac{7}{495}$.
11. Mother has $\frac{\mathbf{5}}{\mathbf{6}}$ yards of cloth. She wants to make napkins
which require $\mathbf{9}$ inches of cloth each. How many napkins can she make? There are $\mathbf{1 2}$ inches in a ft and $\mathbf{3} \mathrm{ft}$ equal 1 yard.
12. A warehouse contains a number of boxes all the same size.
$\frac{\mathbf{2}}{\mathbf{2 1}}$ of the boxes is removed on Monday. $\frac{\mathbf{1}}{\mathbf{6}}$ of the remaining boxes is shipped on Tuesday. $\frac{\mathbf{1}}{\mathbf{2}}$ of the boxes still in the warehouse is trucked out on Wednesday. All of the remaining 190 boxes is sold on Thursday. How many boxes were in the warehouse on Monday?
Hint: Assume you know the number of boxes in the warehouse on Monday. Let that number be locked in the $\boldsymbol{x}$-box.
13. Subtract $\frac{5}{42}-\frac{2}{1155}$
14. Reduce $\left(\frac{\mathbf{1 2}}{\mathbf{3 5}}\right)\left(\frac{\mathbf{2 1}}{4}\right)\left(\frac{\mathbf{1 0}}{\mathbf{9}}\right)\left(\frac{\mathbf{1 1}}{\mathbf{2 6}}\right)$
15. A troupe of boy scouts come to camp with $45 \frac{2}{3} \mathrm{ft}$ of rope. Each scout is to be cut $\mathbf{1} \frac{\mathbf{3}}{4} \mathrm{ft}$ of rope to practice knots. The
amount of rope is just sufficient for all the scouts (a little piece of rope may be left over). How many scouts are in the troupe?
16. Convert $\frac{150}{37}$ to a mixed number.
17. Convert $\mathbf{7} \frac{\mathbf{9}}{\mathbf{2 8}}$ to an improper fraction.
18. Add $\mathbf{2 5} \frac{\mathbf{6}}{\mathbf{7}}+\mathbf{1 2} \frac{\mathbf{1 1}}{\mathbf{1 4}}$.
19. Subtract $38 \frac{\mathbf{9}}{\mathbf{1 4}}-\mathbf{1 2} \frac{\mathbf{1 1}}{14}$.
20. Multiply $\left(3 \frac{6}{7}\right)\left(2 \frac{1}{3}\right)$
21. Convert $\mathbf{8 1} \frac{\mathbf{2}}{\mathbf{3}} \%$ to an exact fraction, no decimals.
22. Convert $\frac{\mathbf{2}}{\mathbf{3}}$ to an exact percent. Give your answer as a mixed number.
23. A retailer buys a dozen coats for $\mathbf{\$ 2 5 0}$ each. The mark up rate (taken as a percent of cost) is $\mathbf{3 0 \%}$. Find the sale price of a coat (cost plus mark up).
A year later the retailer has one coat left and wants to give his customers a discount of $\mathbf{3 0 \%}$ of the sale price to sell the coat. The discount is computed using the sale price above. How much does the customer have to pay after the discount?
Is the discounted price equal to the original cost of $\$ \mathbf{2 5 0}$ since the mark up rate was $\mathbf{3 0 \%}$ and the discount rate was also $\mathbf{3 0 \%}$ ?
If not, what is the retailer's profit or loss?
24. What is $\mathbf{5 \%}$ of $\mathbf{2 5}$ ?
25. $\mathbf{2 5}$ is $\mathbf{5 \%}$ of what number?
26. $\mathbf{5}$ is what percent of $\mathbf{2 5}$ ?

## STOP!

The solutions follow:
Keep them covered up till you have worked out each of the problems above.

1. Find 3 fractions equivalent to $\frac{\mathbf{8}}{\mathbf{1 1}}$ with numerators $\boldsymbol{n} \leq \mathbf{3 5}$.

## Solution:

$$
\begin{aligned}
& \frac{8}{11}=\frac{8}{11} \cdot \frac{2}{2}=\frac{16}{22} \\
& \frac{8}{11}=\frac{8}{11} \cdot \frac{3}{3}=\frac{24}{33} \\
& \frac{8}{11}=\frac{8}{11} \cdot \frac{4}{4}=\frac{32}{44}
\end{aligned}
$$

2. Reduce $\frac{\mathbf{6 3 0}}{1, \mathbf{7 6 4}}$

## Solution:

$$
\begin{aligned}
\frac{630}{1,764} & =\frac{9 \cdot 70}{9 \cdot 196} & \text { Factor } \frac{9}{9}=1 \\
& =\frac{70}{196} & \\
& =\frac{2 \cdot 35}{2 \cdot 98} & \text { Factor } \frac{2}{2}=1 \\
& =\frac{35}{98} & \\
& =\frac{7 \cdot 5}{7 \cdot 14} & \text { Factor } \frac{7}{7}=1 \\
& =\frac{5}{14} &
\end{aligned}
$$

3. Find the units (ones) digit $\boldsymbol{d}$ such that 49,17 " $d$ " is divisible by 2.

## Solution:

$\boldsymbol{d}$ can be $\mathbf{0}$ so the number is $\mathbf{4 9}, \mathbf{1 7 0}$.
$d$ can also be 2 , so the number is 49,172 .
$d$ can further be 4 , so the number is 49,174 .
$\boldsymbol{d}$ can be $\mathbf{6}$ as well, so the number is 49,176 .
Finally $d$ can be 8 , so the number is 49,178 .
4. Find the units (ones) digit $d$ such that 49,17 " $d$ " is divisible by $\mathbf{3}$.

## Solution:


If " $d$ " $=0,21$ is divisible by 3 . The number is 49,170 .
If " $d$ " $=3,24$ is divisible by 3 . The number is 49,173 .
If " $d$ " $=6,27$ is divisible by 3 . The number is 49,176 .
If " $d$ " $=9,30$ is divisible by 3 . The number is 49,179 .
5. If possible, construct a number that is divisible by all of the following numbers: $\mathbf{2 , 3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}$, and 10.

## Solution:

The number must be divisible by $\mathbf{5 , 7 , 8}$, and $\mathbf{9}$.
If the number is divisible by $8=4 \cdot 2$ it will be automatically divisible by 2 and 4 .
If the number is divisible by $\mathbf{9}=\mathbf{3} \cdot \mathbf{3}$ it will be automatically divisible by $\mathbf{3}$.
If the number is divisible by $\mathbf{3}$ and $\mathbf{2}$ it will be automatically divisible by $\mathbf{6}$.
The smallest number we seek is
$N=5 \cdot 7 \cdot 8 \cdot 9=40 \cdot 63=2,520$.
$N$ can be any multiple of $\mathbf{2 , 5 2 0}$, namely $5040,7560,10080,12600,15120, \cdots$
6. Reduce $\left(\frac{\mathbf{1 4}}{35}\right)\left(\frac{36}{77}\right)\left(\frac{49}{24}\right)$

Solution:

$$
\begin{aligned}
\left(\frac{14}{35}\right)\left(\frac{36}{77}\right)\left(\frac{49}{24}\right) & =\left(\frac{2 \cdot 7}{5 \cdot 7}\right)\left(\frac{2 \cdot 2 \cdot 3 \cdot 3}{7 \cdot 11}\right)\left(\frac{7 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 3}\right) \\
& =\frac{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 \cdot 7 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \cdot 7 \cdot 11 \cdot} \\
& =\frac{222 \not 2 \not 2 \cdot 377 \cdot 7}{2222 \not 2 \cdot 577 \cdot 11 \cdot} \\
& =\frac{3 \cdot 7}{5 \cdot 11} \\
& =\frac{21}{55}
\end{aligned}
$$

7. What is the prime factorization of $\mathbf{2 6 4}$ ?

## Solution:

$$
264=2(132)=2(2)(66)=2(2)(2)(33)=2(2)(2)(3)(11)=2^{3}(3)(11)
$$

8. What is the prime factorization of 495 ?

## Solution:

$$
495=(3)(165)=(3)(3)(55)=(3)(3)(5)(11)
$$

9. Find the Least Common Multiple of 264 and 495.

## Solution:

| 264 | $=$ | 2 | 2 | 2 | 3 |  |  | 11 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 495 | $=$ |  |  |  | 3 | 3 | 5 | 11 |
| LCD $=$ | 2 | 2 | 2 | 3 | 3 | 5 | 11 |  |$|=264(3)(5)=264(15)=3,960$

10. Add $\frac{5}{264}+\frac{7}{495}$.

## Solution:

We found that the LCD of $\mathbf{2 6 4}$ and 495 is $\mathbf{3 , 9 6 0}$.
Note from the LCD array above that 264 needs to be multiplied by (3)(5)=15 to get $\mathbf{3 , 9 6 0}$.
Similarly 495 needs to be multiplied by (2)(2)(2) $=8$ to get $\mathbf{3 , 9 6 0}$.

$$
\begin{aligned}
\frac{5}{264} & =\frac{5}{264} \frac{15}{15}
\end{aligned}=\frac{75}{3,960} \quad \text { Quick check: }=0.0189394
$$

11. Mother has $\frac{\mathbf{5}}{\mathbf{6}}$ yards of cloth. She wants to make napkins which require $\mathbf{9}$ inches of cloth each. How many napkins can she make?
There are $\mathbf{1 2}$ inches in a ft and 3 ft equal 1 yard.

## Solution:

For the first napkin,
$\frac{\mathbf{9} \text { inches }}{\mathbf{1}} \cdot \frac{\mathbf{1} \mathrm{ft}}{\mathbf{1 2} \text { inches }}=\frac{\mathbf{3} \cdot \mathbf{3} \mathrm{ft}}{\mathbf{3} \cdot \mathbf{4}}=\frac{\mathbf{3} \mathrm{ft}}{\mathbf{4}}=\frac{\mathbf{3} \mathrm{ft}}{\mathbf{4}} \cdot \frac{\text { yards }}{\mathbf{3} \mathrm{ft}}=\frac{\mathbf{1} \text { yard }}{\mathbf{4}}$
of cloth is removed, cut from the cloth. For the second napkin, $\mathbf{9}$ inches $=\frac{\mathbf{1}}{\mathbf{4}}$ yard of cloth is removed, subtracted, cut from the cloth.
This repeated subtraction suggests division.
The number of napkins is
$\frac{\mathbf{5} \text { yds }}{\mathbf{6}} \div \frac{\mathbf{1} \text { yrd }}{\mathbf{4} \text { napkins }}=\frac{\mathbf{5} \text { yds }}{\mathbf{6}} \cdot \frac{\mathbf{4} \text { napkins }}{\mathbf{1} \text { yrd }}=\frac{\mathbf{5}}{\mathbf{3}} \cdot \frac{\mathbf{2} \text { napkins }}{\mathbf{1}}=\mathbf{3}$ napkins with $\frac{\mathbf{1}}{\mathbf{3}}$ yard of cloth left over.
Instead of using fractions, we could have solved the problem using inches.
One napkin requires 9 inches. The available cloth length is
$\frac{\mathbf{5 ~ y d}}{\mathbf{6}} \cdot \frac{\mathbf{3 ~ f t}}{\mathbf{1} \mathrm{yd}} \cdot \frac{\mathbf{1 2} \mathrm{in}}{\mathbf{1} \mathrm{ft}}=\frac{\mathbf{5}}{\mathbf{6}} \cdot \frac{\mathbf{3}}{\mathbf{1}} \cdot \frac{\mathbf{6} \cdot \mathbf{2} \mathrm{in}}{\mathbf{1}}=\frac{\mathbf{5}}{\mathbf{1}} \cdot \frac{\mathbf{3}}{\mathbf{1}} \cdot \frac{\mathbf{1} \cdot \mathbf{2} \mathrm{in}}{\mathbf{1}}=\mathbf{3 0}$ inches.
The number of napkins is $\frac{\mathbf{3 0} \text { inches }}{\mathbf{1}} \cdot \frac{\mathbf{1} \text { napkin }}{\mathbf{9} \text { inches }}=\frac{\mathbf{3 0}}{\mathbf{9}}=\mathbf{3}$ napkins with $\mathbf{3}$ inches of cloth left over.
12. A warehouse contains a number of boxes all the same size. $\frac{\mathbf{2}}{\mathbf{2 1}}$ of the boxes is removed on Monday. $\frac{\mathbf{1}}{\mathbf{6}}$
of the remaining boxes is shipped on Tuesday. $\frac{\mathbf{1}}{\mathbf{2}}$ of the boxes still in the warehouse is trucked out on Wednesday. All of the remaining 190 boxes is sold on Thursday. How many boxes were in the warehouse on Monday?
Hint: Assume you know the number of boxes in the warehouse on Monday. Let that number be locked in the $\boldsymbol{x}$-box.

## Solution:

Cut the number in the $\boldsymbol{x}$-box into 21 parts and subtract 2 of those parts. That leaves $\mathbf{1}-\frac{\mathbf{2}}{\mathbf{2 1}}=\frac{\mathbf{1 9}}{\mathbf{2 1}}$ of the original number of boxes for Tuesday.

Cut the $\frac{19}{21}$ parts from Monday into six more parts to get
$\frac{19}{21} \cdot \frac{1}{6}=\frac{19}{126}$ parts. $\frac{1}{6}$ of these parts is shipped out on
Tuesday. This leaves $\frac{19}{21}-\frac{19}{126}=\frac{114}{126}-\frac{19}{126}=\frac{95}{126}$ of
the original boxes on Tuesday.
Half of $\frac{\mathbf{9 5}}{\mathbf{1 2 6}} \cdot \frac{\mathbf{1}}{\mathbf{2}}=\frac{\mathbf{9 5}}{\mathbf{2 5 2}}$ the original boxes is sold on
Thursday. That leaves $\frac{\mathbf{9 5}}{\mathbf{2 5 2}}$ of the original boxes for Thursday.
The original number of boxes is $\frac{\mathbf{9 5}}{\mathbf{2 5 2}}$ of the original number of boxes.

The number of boxes is

$$
\frac{252}{95} \cdot 190=\frac{(252)(95)}{95}=\frac{(252)(2)}{1}=504
$$

W0W! You are exceptionally talented if you got the answer on your own by reasoning, not by guessing and trying.

As a preview to algebra, let's rework this problem. Give the name $\boldsymbol{x}$ to the number locked up number in the $\boldsymbol{x}$-box.

$$
x \quad=\frac{\mathbf{2 1}}{21} \text { Original number }
$$

$\frac{\mathbf{2}}{\mathbf{2 1}} \boldsymbol{x} \quad$ Number of boxes sold on Monday
$x-\frac{2}{21} x \quad=\frac{19}{21} x$ Number of boxes left on Monday
$\frac{1}{6} \cdot \frac{19}{21} x \quad=\frac{19}{126} x$ Number of boxes sold on Tuesday
$\frac{19}{21} x-\frac{19}{126} x=\frac{114}{126} x-\frac{19}{126} x=\frac{95}{126} x \quad$ Boxes left on Tuesday
$\frac{1}{2} \cdot \frac{\mathbf{9 5}}{126} x \quad=\frac{\mathbf{9 5}}{\mathbf{2 5 2}} \boldsymbol{x}$ Number of boxes sold on Wednesday
$\frac{95}{126} x-\frac{95}{252} x=\frac{95}{252} x$ Number of boxes left on Wednesday
All of these boxes were sold on Thursday. $\frac{\mathbf{9 5}}{\mathbf{2 5 2}} \boldsymbol{x}=190$ so
$x=\frac{252 \cdot 190}{95}=\frac{252 \cdot 2}{1}=504$
Originally there were 504 boxes in the warehouse.
Let's check this number:
$\frac{2 \cdot 504}{21}=48$ boxes were shipped out on Monday leaving
$504-48=456$ boxes.
$\frac{1 \cdot 456}{6}=76$ boxes were shipped out on Tuesday leaving
$456-76=380$ boxes.
$\frac{\mathbf{1} \cdot \mathbf{3 8 0}}{2}=\mathbf{1 9 0}$ boxes were shipped out on Wednesday
leaving $\mathbf{3 8 0}-\mathbf{1 9 0}=\mathbf{1 9 0}$ boxes.
Does this problem wet your appetite for algebra?
13. Subtract $\frac{5}{42}-\frac{2}{1155}$

## Solution:

A common denominator is needed when adding fractions. We found that the LCD of $\mathbf{4 2}$ and 1155:


not remember whether to use the smallest or the largest exponent.

$$
\begin{aligned}
\frac{5}{42}=\frac{5}{42} \cdot \frac{55}{55} & =\frac{275}{2310} \quad 5 \cdot 11 \text { is missing in row 1. (see LCD array). } \\
\frac{2}{1155}=\frac{2}{1155} \cdot \frac{2}{2} & =\frac{4}{2310} \quad 2 \text { is missing in row } 2 \text { (see LCD array). } \\
& =\frac{275+4}{2310}=\frac{279}{2310}
\end{aligned} \text { Fraction can't be reduced further. }
$$

Quick check:
$\frac{5}{42}=0.11904762$
$\frac{2}{1155}=0.00173160$
$\frac{279}{2310}=0.12077922$
$0.11904762+0.00173160=0.12077922$ (Nice!)
14. Reduce $\left(\frac{12}{35}\right)\left(\frac{21}{4}\right)\left(\frac{10}{9}\right)\left(\frac{11}{26}\right)$

Solution:

$$
\begin{aligned}
\left(\frac{12}{35}\right)\left(\frac{21}{4}\right)\left(\frac{10}{9}\right)\left(\frac{11}{26}\right) & =\left(\frac{2 \cdot 2 \cdot 3}{5 \cdot 7}\right)\left(\frac{3 \cdot 7}{2 \cdot 2}\right)\left(\frac{2 \cdot 5}{3 \cdot 3}\right)\left(\frac{11}{2 \cdot 13}\right) \\
& =\frac{2 \cdot 2 \cdot \not 2 \cdot 3 \cdot \pi \cdot 2 \cdot \not 2 \cdot 11}{5 \cdot \pi \cdot 2 \cdot 2 \cdot \$ \cdot \$ \cdot 2 \cdot 13} \\
& =\frac{11}{13}
\end{aligned}
$$

15. A troupe of boy scouts come to camp with $\mathbf{4 5} \frac{\mathbf{2}}{\mathbf{3}} \mathrm{ft}$ of rope. Each scout is to be cut $\mathbf{1} \frac{\mathbf{3}}{\mathbf{4}} \mathrm{ft}$ of rope to practice knots. The amount of rope is just sufficient for all the scouts (a little piece of rope may be left over). How many scouts are in the troupe?

## Solution:

Subtract $1 \frac{\mathbf{3}}{4}$ ft of rope from $\mathbf{4 5} \frac{\mathbf{2}}{\mathbf{3}}$, then cut and subtract another $1 \frac{\mathbf{3}}{\mathbf{4}} \mathrm{ft}$. Repeated subtraction means division.

The rope can be cut into

$$
\begin{aligned}
45 \frac{2 \mathrm{ft}}{3} \div 1 \frac{3}{4} \frac{\mathrm{ft}}{\mathrm{scout}} & =\frac{(45 \cdot 3+2) \mathrm{ft}}{3} \div \frac{(4 \cdot 1+3)}{4} \frac{\mathrm{ft}}{\text { scout }} \\
& =\frac{137 \mathrm{ft}}{3} \div \frac{7 \mathrm{ft}}{4 \text { scouts }} \\
& =\frac{137 \mathrm{ft}}{3} \cdot \frac{4 \text { scouts }}{7 \mathrm{ft}} \\
& =\frac{137 \cdot 4 \text { scouts }}{3 \cdot 7} \\
& =\frac{548}{21} \text { scouts } \\
& =26 \frac{2}{21} \text { scouts }
\end{aligned}
$$

There are 26 scouts in the troupe.
16. Convert $\frac{150}{37}$ to a mixed number.

Solution:

$$
\frac{150}{37}=4 \frac{2}{37} \quad \begin{array}{r}
47 \begin{array}{|c}
150 \\
-148
\end{array}
\end{array}
$$

17. Convert $\mathbf{7} \frac{\mathbf{9}}{\mathbf{2 8}}$ to an improper fraction.

## Solution:

$$
7 \frac{9}{28}=\frac{7 \cdot 28+9}{28}=\frac{196+9}{28}=\frac{205}{28}
$$

18. Add $25 \frac{6}{7}+\mathbf{1 2} \frac{11}{14}$.

## Solution:

$$
\begin{aligned}
& 25 \frac{6}{7} \\
&+12 \frac{11}{14}=\frac{6}{7}=\frac{6}{7} \cdot \frac{2}{2} \\
& 37=\frac{12}{14} \\
& 0=\frac{11}{14} \\
& 0=\frac{23}{14}=1+\frac{9}{14}
\end{aligned}
$$

Answer: $37+1+\frac{9}{14}=38 \frac{9}{14}$
19. Subtract $\mathbf{3 8} \frac{\mathbf{9}}{\mathbf{1 4}}-\mathbf{1 2} \frac{\mathbf{1 1}}{\mathbf{1 4}}$.

## Solution:

$$
\begin{aligned}
& 38 \frac{9}{14}=38 \quad+\frac{9}{14}=38+\frac{9}{14} \\
& -12 \frac{11}{14}=-\left(12+\frac{11}{14}\right)=-12+\frac{-11}{14} \\
& =26+\frac{-2}{14} \\
& 26+\frac{-2}{14} \\
& =25+1+\frac{-2}{14} \\
& =25+\frac{14}{14}+\frac{-2}{14} \\
& =25+\frac{12}{14} \\
& =25+\frac{2 \cdot 6}{2 \cdot 7} \\
& =25 \frac{6}{7}
\end{aligned}
$$

20. Multiply $\left(\mathbf{3} \frac{6}{7}\right)\left(\mathbf{2} \frac{1}{3}\right)$

Solution:
$\left(3 \frac{6}{7}\right)\left(2 \frac{1}{3}\right)=\left(\frac{27}{7}\right)\left(\frac{7}{3}\right)=\frac{27 \cdot 7}{3 \cdot 7}=\frac{27}{3}=9$
21. Convert $\mathbf{8 1} \frac{\mathbf{2}}{\mathbf{3}} \%$ to an exact fraction, no decimals.

Solution:
$81 \frac{2}{3} \%=\frac{243+2}{3} \%=\frac{245}{3 \cdot 100}=\frac{5 \cdot 49}{3} \cdot \frac{1}{5 \cdot 20}$
$=\frac{49}{3 \cdot 20}=\frac{49}{60}$
22. Convert $\frac{\mathbf{2}}{\mathbf{3}}$ to an exact percent. Give your answer as a mixed number.

## Solution:

$\frac{2}{3}=\frac{2 \cdot 100}{3} \cdot \frac{1}{100}=\frac{200}{3} \%=66 \frac{2}{3} \%$
23. A retailer buys a dozen coats for $\$ \mathbf{2 5 0}$ each. The mark up rate (taken as a percent of cost) is $\mathbf{3 0} \%$. Find the sale price of a coat (cost plus mark up).
A year later the retailer has one coat left and wants to give his customers a discount of $\mathbf{3 0} \%$ of the sale price to sell the coat. The discount is computed using the sale price above. How much does the customer have to pay after the discount?
Is the discounted price equal to the original cost of $\$ \mathbf{2 5 0}$ since the mark up rate was $\mathbf{3 0} \%$ and the discount rate was also $\mathbf{3 0} \%$ ?
If not, what is the retailer's profit or loss?

## Solution:

Mark up is $\mathbf{3 0} \%$ of cost.
Mark up $=(\mathbf{0 . 3})(\mathbf{2 5 0})=\$ 75$
Sale price of a coat $=$ cost plus mark up.
Sale price $=\mathbf{2 5 0}+\mathbf{7 5}=\$ \mathbf{3 2 5}$.
Discount is $\mathbf{3 0} \%$ of sale price.
Discount $=(0.3)(325)=\$ 97.50$
The customer has to pay $\mathbf{3 2 5} \mathbf{- 9 7 . 5 0}=\mathbf{\$ 2 2 7 . 5 0}$.
The retailer loses $\mathbf{2 5 0} \mathbf{- 2 2 7 . 5 0}=\mathbf{\$ 2 2 . 5 0}$ on this last coat.
24 . What is $\mathbf{5} \%$ of $\mathbf{2 5}$ ?

## Solution:

$A=(0.05)(25)$
$A=1.25$
25. $\mathbf{2 5}$ is $\mathbf{5} \%$ of what number?

## Solution:

$25=(0.05) B$
$B=\frac{25}{0.05}$
$B=\frac{2500}{5}=500$
26. $\mathbf{5}$ is what percent of $\mathbf{2 5}$ ?

## Solution:

$5=x(25)$
$x=\frac{5}{25}=\frac{20}{100}=20 \%$

## Chapter 6

## Positive and Negative Real Numbers, the Number Line

### 6.1 Youtube

http://www.youtube.com/playlist?list=PL348B7461EC9415B9\&feature=view_all

### 6.2 The Number Line

Just like there are infinitely many points between any consecutive tick marks (representing integers), there are infinitely many numbers between any two distinct points. Two such rational numbers are $\frac{-3}{\mathbf{2 5}}=-\mathbf{0 . 1 2}$ and $\frac{\mathbf{3}}{\mathbf{7}}=\mathbf{0 . \overline { 4 2 8 5 7 1 }}$, one such irrational number is $\boldsymbol{\pi}=\mathbf{3 . 1 4 1 5 9} \cdots$. These numbers can be written as decimal numbers.


Let's review the "boxes" (places) which we use to collect objects and group them in the decimal system. Let's start with some place like one thousand or $\mathbf{1 0}^{\mathbf{3}}=\mathbf{1 , 0 0 0}$.

| thousand | hundred | ten | one | tenth | hundredth | thousandth |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\cdots \mathbf{1 , 0 0 0}$ | $\frac{\mathbf{1 0 0 0}}{\mathbf{1 0}}=\mathbf{1 0 0}$ | $\frac{\mathbf{1 0 0}}{\mathbf{1 0}}=\mathbf{1 0}$ | $\frac{\mathbf{1 0}}{\mathbf{1 0}}=\mathbf{1}$ | $\frac{\mathbf{1}}{\mathbf{1 0}}$ | $\frac{\mathbf{1}}{\mathbf{1 0 0}}$ | $\frac{\mathbf{1}}{\mathbf{1 , 0 0 0}} \cdots$ |
| $\mathbf{1 0}^{\mathbf{3}}$ | $\mathbf{1 0}^{\mathbf{2}}$ | $\mathbf{1 0}^{\mathbf{1}}$ | $\mathbf{1 0}^{\mathbf{0}}$ | $\mathbf{1 0}^{-\mathbf{1}}$ | $\mathbf{1 0}^{-\mathbf{2}}$ | $\mathbf{1 0}^{-\mathbf{3}}$ |

We are extending the boxes (places) to the right of the ones (after the double vertical bars in the array above.)

It is convenient to use a marker to identify the digit in the ones place.
123.45 is such a number. The digit $\mathbf{3}$ is in the ones place.
1.2345 is another such number. The digit $\mathbf{1}$ is in the ones place.
$\mathbf{0 . 0 1 2 3 4 5}$ is still another such a number. The first $\mathbf{0}$ is in the ones place. The digit $\mathbf{2}$ is in the thousandth place.

The decimal point is the ones digit marker in America (it is a comma in much of the world).
Write 123.45 in words: one hundred twenty-three and $\mathbf{4 5}$ hundredths.
Note that "and" translates into the decimal point.
Also note that the set of digit after the decimal point is considered one group. The name of the group is the place (box) of the rightmost digit of the set, like $\mathbf{4 5}$ hundredths.
1.2345 is one and two thousand three hundred forty-five ten-thousandths.
0.012345 is twelve thousand three hundred forty-five millionths.

### 6.3 Addition of decimals

| What is wrong with $\quad \begin{array}{r}24.6 \\ \hline\end{array} \quad \begin{array}{r}9 \\ \hline\end{array} \quad 1 \quad 3.58$ |
| :--- |
| 116.028 |

We are adding apples to oranges! For example, $\mathbf{9}$ "hundreds" is added $\mathbf{2}$ "tens".

$$
\begin{array}{r}
244.67 \\
+913.58 \\
\hline 938.25
\end{array}
$$

What is new in adding decimal numbers? Nothing beyond lining up the decimal point. This ensures that we add numbers of the the same place values (boxes).

### 6.4 Subtraction of Decimals

What is new in subtracting decimal numbers? Nothing beyond lining up the decimal point. This ensures that we add numbers of the the same place values (boxes).

### 6.5 Multiplication of Decimals



|  | 1 | 2. | 3 |
| ---: | ---: | ---: | ---: |
| $\times$ | 5 | 1 |  |
|  | 1 | 2 | 3 |
| 6 | 1 | 5 |  |
| 6 | 2 | 7. | 3 |


|  | 1. | 2 | 3 |
| ---: | ---: | ---: | ---: |
| $\times$ |  | 5 | 1 |
|  | 1 | 2 | 3 |
| 6 | 1 | 5 |  |
| 6 | 2. | 7 | 3 |

$1.20 \times 50=60$
The factor $\mathbf{1 . 2 3}$ has two decimals, the product 62.73 also has two decimals.

|  | 1. | 2 | 3 |
| ---: | ---: | ---: | ---: |
| $\times$ |  | 5. | 1 |
|  | 1 | 2 | 3 |
| 6 | 1 | 5 |  |
| 6. | 2 | 7 | 3 |

$12.0 \times 50=600$
The factor 12.3 has one decimal, the product 627.3 also has one decimal.

## Solution:

$\left(6.7 \times 10^{4}\right)\left(5.1 \times 10^{3}\right)=(34.17)\left(10^{4+3}\right)=3.417 \times 10^{8}$

### 6.6 Multiplication/division with powers of 10

We are using the decimal system of counting. Multiplication is repeated addition. These two facts combine to justify the following statement:

Multiplication by $\mathbf{1 0}$ is equivalent to moving the decimal point one place to the right.

Division by $\mathbf{1 0}$ is equivalent to moving the decimal point one place to the left.

Is there a decimal point in the whole number $\mathbf{2}$ ?
Yes, there is. It is just invisible. $\mathbf{2}=\mathbf{2 . 0}$. The first number to the left of the decimal point is in the ones place.
$(2)(10)=(2.0)(10)$
$=2.0+2.0+2.0+2.0+2.0+2.0+2.0+2.0+2.0+2.0=20.0$
(1) (2) (3) (4) (5) (6) (7) (8) (9) (10)

Division is repeated subtraction. Reverse the reasoning above. Subtract $\mathbf{2 . 0}$ ten times from $\mathbf{2 0 . 0}$ to get $\mathbf{2 . 0}$.
Convert numbers from scientific (or exponential) notation to standard (or decimal) notation.
$3.17 \times 10^{5}=317,000$ and $3.17 \times 10^{-5}=0.0000317$

### 6.7 Division of Decimals


$5 \quad 1$


| -5 | 1 |  |
| ---: | ---: | ---: |
| 1 | 1 | 7 |


| -1 | 0 | 2 |  |
| ---: | ---: | ---: | ---: |
|  | 1 | 5 | 3 |
|  | -1 | 5 | 3 |

Move the decimal point up from the dividend (the number divided into) to the quotient (the result of the division), in the same vertical position.

$$
\frac{60.00}{50}=1.20 \quad \frac{600.0}{50}=12.0
$$



Move the decimal point up from the dividend (the number divided into) to the quotient (the result of the division), in the same vertical position.


Move the decimal point in the divisor (denominator) as many places as needed to remove it. Move the decimal point in the numerator the same number of places (even if the decimal point is remaining). This is permissible because we multiply the original fraction (division) by $\mathbf{1}=\frac{\mathbf{1 0}}{\mathbf{1 0}}=\frac{\mathbf{1 0 0}}{\mathbf{1 0 0}}=\frac{\mathbf{1 0 0 0}}{\mathbf{1 , 0 0 0}}=\ldots$

### 6.8 Exercise 6

1. Construct an irrational number $N$ so that $\mathbf{3 . 1}<\boldsymbol{N}<\mathbf{3 . 2}$ (the answer is not unique).
2. Write $\mathbf{4 5 . 0 0 4 5}$ in words.
3. Find the sum of $\mathbf{5 . 7 1 5}$ and $\mathbf{- 3 6 . 2 9}$.
4. A $\mathbf{\$ 4 5 . 9 9}$ dress is marked $\mathbf{3 0 \%}$ off. You pay a sales tax of $\mathbf{5 \%}$ on the discounted sales price. How much do you have to pay?
5. What number is subtracted from 4.7 to get $\mathbf{7 . 4}$ ?
6. Multiply $1.35 \cdot(-\mathbf{0 . 0 9})$ and write the product in scientific notation.
7. Divide $\mathbf{0 . 0 1 7 6 6 1} \div \mathbf{2 . 0 3}$
8. Find the product of $\mathbf{1}, \mathbf{0 0 0}$ and $\mathbf{2 . 5 1}$.
9. Find the quotient of $\mathbf{2 . 5 1}$ and $\mathbf{1}, \mathbf{0 0 0}$.
10. Add $\mathbf{3 . 6 2} \times \mathbf{1 0}^{\mathbf{4}}$ to $\mathbf{4 . 1 5} \times \mathbf{1 0}^{\mathbf{3}}$ and give the sum in scientific notation.
11. Find the product of $\mathbf{3 . 6 2} \times \mathbf{1 0}^{\mathbf{4}}$ and $\mathbf{4 . 1 5} \times \mathbf{1 0}^{\mathbf{3}}$. Give the product in scientific notation.
12. Find the quotient $\left(\mathbf{1 . 2} \times \mathbf{1 0}^{\mathbf{5}}\right) \div\left(\mathbf{3 . 6} \times \mathbf{1 0}^{-\mathbf{3}}\right)$ Give the quotient in scientific notation.
13. A rectangular kitchen measures $\mathbf{2 2} \times \mathbf{1 7} \mathrm{ft}$. The floor is to be tiled, except for a rectangular corner $\mathbf{5} \times \mathbf{3} \mathrm{ft}$ for the sink and washing machine. Each tile is a square $\mathbf{1} \times \mathbf{1} \mathrm{ft}^{\mathbf{2}}$.
(a) How many tiles are needed?

Tiles are sold in boxes of $\mathbf{1 5}$ each. How many boxes are to be bought?
The price per box is $\$ \mathbf{2 3 . 9 9}$.
(b) What is the price of the boxes?

The sakes tax on purchases is $\mathbf{9 . 7 5 \%}$.
(c) How much do you have to pay to get the tiles out of the store?

A handyman charges $\mathbf{\$ 2 5}$ per hour to lay the tiles. He takes hours $\mathbf{3 6}$ minutes to lay the tiles.
(d)How much will he get paid (round to the nearest dollar.)
(e) How much money will you have to borrow through a bank for the project? (There is no down payment.)

## Stop!!!

### 6.9 Solutions

1. Construct an irrational number $\boldsymbol{N}$ so that $\mathbf{3 . 1}<\boldsymbol{N}<\mathbf{3 . 2}$ (the answer is not unique).

## Solution:

Append $01011011101111 \cdots$ to 3.1 to get 3.101011011101111 ...
2. Write $\mathbf{4 5 . 0 0 4 5}$ in words.

## Solution:

Forty-five and forty-five ten-thousandths.
3. Find the sum of $\mathbf{5 . 7 1 5}$ and $\mathbf{- 3 6 . 2 9}$.

## Solution:

$$
\begin{aligned}
& 5.715+(-36.29)=-(36.29-5.715) \\
& 36 \quad . \quad 290 \\
& \begin{array}{rrrrr}
- & 5 & 7 & 1 & 5 \\
\hline & 3 & 0 & 5 & 7
\end{array}
\end{aligned}
$$

The sum of $\mathbf{5 . 7 1 5}$ and $-\mathbf{3 6 . 2 9}$ is $\mathbf{- 3 0 . 5 7 5}$
4. A $\$ \mathbf{4 5 . 9 9}$ dress is marked $\mathbf{3 0 \%}$ off. You pay a sales tax of $\mathbf{5 \%}$ on the discounted sales price. How much do you have to pay?

## Solution:

The discount is $\mathbf{3 0 \%}$ of $\mathbf{\$ 4 5 . 9 9}$
$D=(\mathbf{0 . 3})(\mathbf{4 5 . 9 9})=\mathbf{1 3 . 8 0}$ (rounded to the nearest cent).
The sales price is $\mathbf{4 5 . 9 9 - 1 3 . 8 0 = 3 2 . 1 9}$
The sales tax is $\mathbf{5 \%}$ of the discounted sales price.
$T=(0.05)(32.19)=1.61$
You will have to pay $32.19+1.61=\$ 33.80$
5. What number is subtracted from 4.7 to get $\mathbf{7 . 4}$ ?

## Solution:

Let $N$ be the number. Then $4.7-N=7.4$ or $N=4.7-7.4=-(7.4-4.7)=-2.7$
6. Multiply $\mathbf{1 . 3 5} \cdot(-\mathbf{0 . 0 9})$ and write the product in scientific notation.

## Solution:

1

1 21 | 3 |
| :--- |

so
$(\mathbf{1 . 3 5})(-\mathbf{0 . 0 9})=-\mathbf{0 . 1 2 1 5}$ (There are four decimal places total.)
I do not recommend using $\mathbf{1 . 3 5} \cdot \mathbf{0 . 0 9}$ for multiplication because of the possible confusion between the decimal point "." (a period!) and the multiplication sign ".".
I especially object to $1.35 \cdot-.9$ (Did you mean subtraction and there a fly speck on the paper?)
Other ways of writing multiplication: $\mathbf{5} \times \mathbf{3}$ (if the multiplication sign is not confused with the
variable $x)(5)(3), 5(3)$
Warning: $\mathbf{5}-\mathbf{3}$ is $\mathbf{5}$ minus $\mathbf{3}$ while $\mathbf{5}(-\mathbf{3})$ means $\mathbf{5}$ times $\mathbf{- 3}$.
7. Divide $\mathbf{0 . 0 1 7 6 6 1} \div \mathbf{2 . 0 3}=\mathbf{0 . 0 0 8 7}$

## Solution:

$0.017661 \div 2.03=\frac{0.017661}{2.03}=\frac{1.7661}{203}$

8. Find the product of $\mathbf{1}, \mathbf{0 0 0}$ and $\mathbf{2 . 5 1}$.

## Solution:

$$
2.51(1,000)=2,510 \quad(1,000=(10)(10)(10))
$$

9. Find the quotient of $\mathbf{2 . 5 1}$ and $\mathbf{1 , 0 0 0}$.

## Solution:

$2.51 \div 1,000=\frac{2.51}{1,000}=0.00251$
The product of $(\mathbf{2} .51)(\mathbf{0 . 0 0 1})=$ the quotient of $\mathbf{2 . 5 1} \div \mathbf{0 . 0 0 1}$.
10. Add $\mathbf{3 . 6 2} \times \mathbf{1 0}^{\mathbf{4}}$ to $\mathbf{4 . 1 5} \times \mathbf{1 0}^{\mathbf{3}}$ and give the sum in scientific notation.

## Solution:

$$
\begin{array}{rl}
3.62 \times 10^{4} & = \\
3 & 6 \\
2 & 0 \\
0 \\
+\quad 4.15 \times 10^{3} & = \\
4 & 1 \\
& =40
\end{array} \mathbf{5} 5000=4.035 \times 10^{4}
$$

11. Find the product of $\mathbf{3 . 6 2} \times \mathbf{1 0}^{\mathbf{4}}$ and $\mathbf{4 . 1 5} \times \mathbf{1 0}^{\mathbf{3}}$. Give the product in scientific notation.

## Solution:



Converting numbers to standard (decimal) form first leads to

12. Find the quotient $\left(\mathbf{1 . 2} \times \mathbf{1 0}^{\mathbf{5}}\right) \div\left(\mathbf{3 . 6} \times \mathbf{1 0}^{\mathbf{- 3}}\right)$ Give the quotient in scientific notation.

## Solution:

$$
\begin{aligned}
\left(1.2 \times 10^{5}\right) \div\left(3.6 \times 10^{-3}\right) & =\frac{1.2 \times 10^{5}}{3.6 \times 10^{-3}} \\
& =\frac{1.2}{3.6} \times 10^{5-(-3))} \\
& =\frac{1}{3} \times 10^{8}=0 . \overline{3} \times 10^{8} \\
& =(0 . \overline{3} \times 10)\left(\times 10^{7}\right) \\
& =3 . \overline{3} \times 10^{7}
\end{aligned}
$$

13. A rectangular kitchen measures $22 \times$ 17 ft . The floor is to be tiled, except for a rectangular corner $\mathbf{5} \times \mathbf{3} \mathrm{ft}$ for the sink and washing machine. Each tile is a square $\mathbf{1} \times \mathbf{1} \mathrm{ft}$.

(a) How many tiles are needed? Tiles are sold in boxes of $\mathbf{1 5}$ each. How many boxes are to be bought?
The price per box is $\mathbf{\$ 2 3 . 9 9}$.
(b) What is the price of the boxes?

The sales tax on purchases is $\mathbf{9 . 7 5 \%}$.
(c) How much do you pay to get the tiles out of the store?

A handyman charges $\mathbf{\$ 2 5}$ per hour to lay the tiles. He takes $\mathbf{5}$ hours $\mathbf{3 6}$ minutes to lay the tiles.
(d)How much will he get paid (round to the nearest \$.)
(e) How much money will you have to borrow through a bank for the project? (There is no down payment.)

## Solution:

The area of the rectangular kitchen is $\mathbf{( 2 2 ) ( 1 7 )}=\mathbf{3 7 4} \mathrm{ft}^{2}$.
The area for the sink and washing machine is $\mathbf{3} \times \mathbf{5}=15 \mathrm{ft}^{2}$.
The area to be tiled is $\mathbf{3 7 4 - 1 5}=\mathbf{3 5 9} \mathrm{ft}^{\mathbf{2}} . \mathbf{3 5 9}$ tiles are needed.
After gluing 15 tiles, one box has been consumed (subtracted). Subtract a second box for the next $\mathbf{1 5}$ tiles. The number of tiles is $\frac{\mathbf{3 5 9} \text { tiles }}{\mathbf{1}} \cdot \frac{\text { box }}{\mathbf{1 5} \text { tiles }}=\mathbf{2 3 . 9}$ boxes.
(a) $\mathbf{2 4}$ boxes need to be bought.
(b) The cost is $\frac{\mathbf{2 3 . 9 9} \text { dollars }}{\text { box }} \cdot \frac{\mathbf{2 4} \text { boxes }}{\mathbf{1}}=\$ 575.76$.
(c) The sales tax is $\mathbf{0 . 0 9 7 5} \cdot \mathbf{5 7 5 . 7 6}=\mathbf{5 6 . 1 4}$
(d) $575.76+\mathbf{5 6 . 1 4}=\$ 631.90$

5 hours and $\mathbf{3 6}$ minutes is $\mathbf{5} \frac{\mathbf{3 6}}{60}=5 \frac{3}{5}=5.6$ hours. The
wages for the handyman are $\frac{\mathbf{5 . 6} \text { hours }}{\mathbf{1}} \cdot \frac{\mathbf{2 5} \text { dollars }}{\text { hour }}=\$ \mathbf{1 4 0}$
(e) The total project costs $\mathbf{5 7 5 . 7 6}+\mathbf{1 4 0}=\$ \mathbf{7 1 5 . 7 6}$, which is the amount to be borrowed.

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## Chapter 7

## Order of Operations.

### 7.1 Youtube

http://www.youtube.com/playlist?list=PL72BAD92F11B75FAF\&feature=view_all
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You are paid $\$ 2+\mathbf{3} \cdot \mathbf{4}$ per hour for a job that takes $\mathbf{2 0}$ hours to complete. Your employer wants to pay you $\$ \mathbf{1 4}$ per hour but you feel cheated. You think you earned $\$ \mathbf{2 0}$. Who is right?
$\$ 2+3 \cdot 4=2+12=14$ or $\$ 2+3 \cdot 4=5 \cdot 4=20$ ? In order to avoid confusion like this, smart people have agreed on the following order of operations:

1. If there is an innermost group (like $\mathbf{P}$ arentheses), perform the following in that group:
2. Carry out Exponentiation (if applicable) in that group.
3. Perform $\mathbf{M u l t i p l i c a t i o n ~ a n d / o r ~} \mathbf{D}$ ivision (in the order of occurrence from left to right), if any, in that group.
4. Perform $\mathbf{A d d i t i o n}$ and/or $\mathbf{S}$ ubtraction (in the order of occurrence from left to right), if any, in that group.
You are now down to one number (in arithmetic). Repeat steps $\mathbf{1 - 4}$ in the next innermost group. If there is no innermost group, the entire expression is the group.

The term "Parentheses" above is misleading. It could refer to an expression inside parentheses, brackets, braces, under a square root sign, between absolute values, a numerator, a denominator, ....

Example 1:
Simplify $4+5\left[3^{2}\left(4^{2} \div 2^{3} \cdot 3-5-1\right)+7\right]$

## Solution:

```
    4+5[3'(4}\mp@subsup{4}{}{2}\div\mp@subsup{2}{}{3}\cdot3-5-1)+7] Copy every unmodified symbol.
    PEMDAS - innermost parentheses.
= 4+5[3'(16\div8\cdot3-5-1)+7] Exponentiation PEMDAS
= 4+5[3'2 (2, 3-5-1) + 7] Division precedes multiplication.PEMDAS
= 4+5[3'(6-5-1)+7] Multiply PEMDAS
= 4+5[3'(1-1)+7] Subtract. PEMDAS
= 4+5[3'(0)+7] New group. PEMDAS. Exponentiation.
= 4+5[9(0) + 7] Multiplication. PEMDAS
= 4+5[0+7] Addition. PEMDAS
= 4+5[7] New Group - the entire expression - Multiplication. PEMDAS
= 4+35 Addition. PEMDAS
= 39
```

Example 2:
Simplify $\frac{15+50 \div 5 \cdot 3}{50+15 \div(3 \cdot 5)-6}$

## Solution:

$$
\begin{aligned}
& \frac{15+50 \div 5 \cdot 3}{50+15 \div(3 \cdot 5)-6} \\
= & \frac{\text { pemDas }}{\text { peMdas }} \\
= & \frac{15+10 \cdot 3}{50+15 \div(15)-6} \\
= & \frac{\text { peMdas }}{\text { pemDas }} \\
= & \frac{45}{50+1-6} \\
= & \frac{\text { pemdAs }}{45}=1
\end{aligned}
$$

## Example 3:

If you get the correct answer for the next example the first time, without help from any source, you are exceptionally good. Most students make at least one of three common mistakes. Copy all unmodified symbols vertically. Do scratch work on the right.

Simplify $8-5\left(2^{7} \div 2^{4} \cdot 2^{3}+6-5 \cdot 3-15 \div 3-20-10-5\right)$

## Solution:

|  | $8-5\left(2^{7} \div 2^{4} \cdot 2^{3}+6-5 \cdot 3-15 \div 3-20-10-5\right)$ |
| :--- | :--- |
| $=8-5(128 \div 16 \cdot 8+6-5 \cdot 3-15 \div 3-20-10-5)$ | $2^{3}=8,2^{4}=16$, |
| $=8-5(8 \cdot 8+6-5 \cdot 3-5-20-10-5)$ | $2^{7}=\left(2^{3}\right)\left(2^{4}\right)=(8)(16)=128$. |
| $=8-5(64+6-15-5-20-10-5)$ | Division first. |
| $=8-5(70-15-5-20-10-5)$ | Multiplication next. |
| $=8-5(55-5-20-10-5)$ | Addition. |
| $=8-5(50-20-10-5)$ | Subtraction. |
| $=8-5(30-10-5)$ | Subtraction. |
| $=8-5(20-5)$ | Subtraction. |
| $=8-5(15)$ | Subtraction. |
| $=8-75$ | Subtraction. |
| $=-67$ | Subtraction. |

Note 1:

$$
2^{7} \div 2^{4} \cdot 2^{3}=\frac{2^{7}}{2^{4}} \cdot 2^{3}=\frac{2 \cdot 2 \cdot 22 \cdot 22 \cdot 2 \cdot 2 \cdot 2}{\cdot 22 \cdot 22 \cdot 2} \cdot 2^{3}
$$

$=2^{4} \cdot 2^{3}=2^{4+3}=2^{7}$
Note 2: First common mistake: $\mathbf{8 - 5}=\mathbf{3}$ (not in this example!)
Second common mistake:
$2^{7} \div 2^{4} \cdot 2^{3}=2^{7} \div\left(2^{4} \cdot 2^{3}\right)=2^{7} \div 2^{7}=1$ WRONG. The offender belongs to the IPS (Invisible Parentheses Society) and needs to give up his/her membership immediately.

Third common mistake:
$6-5=1$ (not here) $\mathbf{3 - 1 5}=-12$ (not here) $\mathbf{3 - 2 0}=-\mathbf{1 7}$ (not here) $20-10=10$ (not here) $15-5=10$.

Note 3: You will be able to take shortcuts once you become proficient. If you have doubts, follow the order of operations as outlined here.

### 7.2 Exercises 7

1. Simplify $\mathbf{4 - 3 \cdot 1 5}$
2. Simplify $4-3 \cdot 15 \div 3$
3. Simplify $4-81 \div 3 \div 3$
4. Simplify $(4-15) \cdot 3 \div 3$
5. Simplify $\frac{-18 \cdot 2+9}{-(-18)-(2+16)}$
6. Simplify $5+3^{2}\left[2^{9} \div 2^{7} \cdot 2^{4}+2 \cdot 3-2\left(5^{2}-2^{5}\right)-30-40\right]$

## STOP!

1. Simplify $4-\mathbf{3} \cdot \mathbf{1 5}$

Solution:

$$
4-3 \cdot 15=4-45
$$

$$
=-41
$$

2. Simplify $4-3 \cdot 15 \div 3$

Solution:

$$
\begin{aligned}
4-3 \cdot 15 \div 3 & =4-45 \div 3 \\
& =4-15 \\
& =-11
\end{aligned}
$$

3. Simplify $4-81 \div 3 \div 3$

$$
\begin{aligned}
& \text { Solution: } \\
& \begin{aligned}
4-81 \div 3 \div 3 & =4-27 \div 3 \\
& =4-9 \\
& =-5
\end{aligned}
\end{aligned}
$$

4. Simplify $(4-15) \cdot 3 \div 3$

## Solution:

$$
\begin{aligned}
(4-15) \cdot 3 \div 3 & =(-11) \cdot 3 \div 3 \\
& =-33 \div 3 \\
& =-11
\end{aligned}
$$

5. Simplify $\frac{-18 \cdot 2+16}{-(-18)-(\mathbf{2}+\mathbf{9})}$

Solution:

$$
\begin{aligned}
\frac{-18 \cdot 2+9}{-(-18)-(2+16)} & =\frac{-36+9}{-(-18)-(18)} \\
& =\frac{-27}{18-18} \\
& =\frac{-27}{0}
\end{aligned}
$$

The operation of dividing by 0 is undefined. (Do not use $\emptyset$ )
6. Simplify $5+3^{2}\left[2^{9} \div 2^{7} \cdot 2^{4}+2 \cdot 3-2\left(5^{2}-2^{5}\right)-30-40\right]$

Solution:

$$
\begin{aligned}
& \text { Solution: } \\
& \begin{aligned}
& 5+3^{2}\left[2^{9} \div 2^{7} \cdot 2^{4}+2 \cdot 3-2\left(5^{2}-2^{5}\right)-30-40\right] \\
= & 5+3^{2}\left[2^{9} \div 2^{7} \cdot 2^{4}+2 \cdot 3-2(25-32)-30-40\right] \\
= & 5+3^{2}\left[2^{9} \div 2^{7} \cdot 2^{4}+2 \cdot 3-2(-7)-30-40\right] \\
= & 5+3^{2}[512 \div 128 \cdot 16+2 \cdot 3-2(-7)-30-40] \\
= & 5+3^{2}[4 \cdot 16+6-2(-7)-30-40] \\
= & 5+3^{2}[64+6-(-14)-30-40] \\
= & 5+3^{2}[64+6+14-30-40] \\
= & 5+3^{2}[70+14-30-40] \\
= & 5+3^{2}[84-30-40] \\
= & 5+3^{2}[54-40] \\
= & 5+3^{2}[14] \\
= & 5+9[14] \\
= & 5+126 \\
= & 131
\end{aligned}
\end{aligned}
$$

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## Chapter 8

## Evaluating Expressions

### 8.1 Youtube

http://www.youtube.com/playlist?list=PL035D120A0C52BDD9\&feature=view_all
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### 8.2 Definition of a Variable Expression

A variable expression is a collection of numbers, letters (variables), operations, grouping symbols, any mathematical symbol except an equal sign or an inequality sign.

Pieces of an expression related by + or - are terms. Pieces of an expression related through a multiplication sign are factors.

Examples of expressions:
$2 a+5$
$3 x^{2}-4 y$
$\frac{x^{3}-y^{3}}{x-y}$
$V-\pi x^{2} y$
$(a+b)^{2}-a^{2}-b^{2}$
$A-\frac{h}{2(B+b)}$
A rational expression is a ratio (fraction) of two expressions. The term is usually reserved for the ratio of
two polynomials (to be defined in another chapter.)
$\frac{x^{2}+6 x y+5 y^{2}}{x+2 y}$

### 8.3 Evaluation of Expressions

To evaluate means to find the value of something. Evaluating $2 \boldsymbol{a}$ if $\boldsymbol{a}=\mathbf{5}$ means finding the value of twice the number in the "a"-box which is known (in this example to be $\mathbf{5}$. Open a set of parentheses in place of the variable $\boldsymbol{a}$, then drop the value of $\boldsymbol{a}=\mathbf{2}$ into these parentheses. Thus $2 \boldsymbol{a}=\mathbf{2 ( 5 )}=\mathbf{1 0}$.

Example 1:
Evaluate $2 a+5$ if $\boldsymbol{a}=\mathbf{- 3}$.

## Solution:

$$
\begin{aligned}
2 a+5 & =2()+5 \\
& =2(-3)+5 \\
& =-6+5 \\
& =-1
\end{aligned}
$$

Example 2:
Evaluate $3 x^{2}-4(y-3)$ if $x=-2$ and $y=-3$.

## Solution:

$$
\begin{aligned}
3 x^{2}-4(y-3) & =3()^{2}-4[()-3] \\
& =3(-2)^{2}-4[(-3)-3] \\
& =3(4)-4(-3-3) \\
& =12-4(-6) \\
& =12+24 \\
& =36
\end{aligned}
$$

Example 3:
Evaluate $\frac{x^{3}-y^{3}}{x-2 y}$ if $x=4$ and $y=2$.

## Solution:

$$
\begin{aligned}
\frac{x^{3}-y^{3}}{x-2 y} & =\frac{()^{3}-()^{3}}{()-2()} \\
& =\frac{(4)^{3}-(2)^{3}}{(4)-2(2)} \\
& =\frac{64-8}{4-4} \\
& =\frac{56}{0} \text { which is undefined }
\end{aligned}
$$

Example 4:
Evaluate $V-\boldsymbol{\pi} \boldsymbol{x}^{2} \boldsymbol{y}$ if $\boldsymbol{V}=\mathbf{3 , 1 4 0}, \boldsymbol{x}=10$ and $\boldsymbol{y}=3$. Approximate $\boldsymbol{\pi}=\mathbf{3 . 1 4}$.

## Solution:

$$
\begin{aligned}
V-\pi x^{2} y & =()-()()^{2}() \\
& =(3,140)-(3.14)(10)^{2}(3) \\
& =3,140-(3.14)(100)(3) \\
& =3,140-(314)(3) \\
& =3,140-942 \\
& =2,198
\end{aligned}
$$

Example 5:
Evaluate $(a+b)^{2}-a^{2}-b^{2}$ if $a=6$ and $b=-6$.
Solution:

$$
\begin{aligned}
(a+b)^{2}-a^{2}-b^{2} & =[()+()]^{2}-()^{2}-()^{2} \\
& =[(6)+(-6)]^{2}-(6)^{2}-(-6)^{2} \\
& =(0)^{2}-36-36=-72
\end{aligned}
$$

Example 6:
Evaluate $A-\frac{h}{2(B+b)}$ if $A=100, h=40, B=12$ and $b=8$.

## Solution:

$$
\begin{aligned}
A-\frac{h}{2(B+b)} & =()-\frac{()}{2[()+()]} \\
& =(100)-\frac{(40)}{2[(12)+(8)]} \\
& =100-\frac{40}{2(20)} \\
& =100-\frac{2}{2(1)} \\
& =100-1=99
\end{aligned}
$$

Example 7:
Evaluate $\frac{x^{2}+6 x y+5 y^{2}+1.99}{x+2 y}$ if $x=0.4$ and $y=1.5$.

## Solution:

$$
\begin{aligned}
\frac{x^{2}+6 x y+5 y^{2}+1.99}{x+2 y} & =\frac{()^{2}+6()()+5()^{2}+1.99}{()+2()} \\
& =\frac{(0.4)^{2}+6(0.4)(1.5)+5(1.5)^{2}+1.99}{(0.4)+2(1.5)} \\
& =\frac{0.16+(2.4)(1.5)+5(2.25)+1.99}{0.4+3} \\
& =\frac{0.16+3.6+11.25+1.99}{3.4}=\frac{3.76+11.25+1.99}{3.4} \\
& =\frac{15.01+1.99}{3.4}=\frac{17}{3.4}=\frac{170}{34}=5
\end{aligned}
$$

Example 8:
Evaluate $4 x^{4}-x^{2}+\frac{7}{9}$ if $x=\frac{1}{2}$.
Solution:

$$
\begin{aligned}
4 x^{4}-x^{2}+\frac{7}{9} & =4()^{4}-()^{2}+\frac{7}{9} \\
& =4\left(\frac{1}{2}\right)^{4}-\left(\frac{1}{2}\right)^{2}+\frac{7}{9} \\
& =4\left(\frac{1}{16}\right)-\frac{1}{4}+\frac{7}{9} \\
& =\frac{1}{4}-\frac{1}{4}+\frac{7}{9} \\
& =\frac{7}{9}
\end{aligned}
$$

Example 9:
Evaluate $y-\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right)$ if $x=4, x_{1}=-2, x_{2}=-5, y=10 y_{1}=9$ and $y_{2}=7$.
Solution:

$$
\begin{aligned}
y-\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right) & =()-\frac{()-()}{()-()}[()-()] \\
& =(10)-\frac{(7)-(9)}{(-5)-(-2)}[(4)-(-2)] \\
& =10-\frac{-2}{-3}(6) \\
& =10-\frac{-2}{1}(-2) \\
& =10-(-2)(-2) \\
& =10-4 \\
& =6
\end{aligned}
$$

Example 10:
A logging company cut a certain number (say T) of trees on Monday. On Tuesday the company cut 5 more trees than on Monday. On Wednesday the number of trees harvested was twice the number of Tuesday. On Thursday the number was half of the number on Monday.
(a) Write an expression for the total number of trees cut on the four days.
(b) If 22 trees were cut down on Monday, what was the total number of trees harvested on the four days?

## Solution:

| Trees cut on Monday | $\boldsymbol{T}$ |
| :--- | ---: |
| Trees cut on Tuesday | $\boldsymbol{T}+\mathbf{5}$ |
| Trees cut on Wednesday | $\mathbf{2 ( T + \mathbf { 5 } )}$ |
| Trees cut on Thursday | $\frac{\boldsymbol{T}}{\mathbf{2}}$ |

(a) Total number of trees cut: $T+(T+5)+2(T+5)+\frac{T}{2}$
or $4 T+\frac{T}{2}+15=\frac{9 T}{2}+15$
(b) Evaluate if $\boldsymbol{T}=\mathbf{2 2}$ :

$$
\begin{aligned}
& \frac{9 \cdot 22}{2}+15 \\
= & \frac{9 \cdot 22}{2}+15 \\
= & 9 \cdot 11+15 \\
= & 99+15 \\
= & 114 \text { trees. }
\end{aligned}
$$

Example 11:
Abel works 14 hours in a particular week. Bianca works 12 hours in that week. Abel gets paid $\$ \boldsymbol{a}$ per hour and Bianca earns $\$ \boldsymbol{b}$ per hour.
(a) Write an expression for the total wages Abel and Bianca earn in that week.
(b) Then evaluate that expression if Abel gets $\$ 8$ per hour and Bianca earns $\$ 12$ per hour.

## Solution:

(a) Abel and Bianca earn $\mathbf{1 4 a}+\mathbf{1 2 b}$ dollars in that week.
(b) Evaluating $14 a+12 b$ leads to

$$
\begin{aligned}
14()+12() & =14(8)+12(12) \\
& =112+144 \\
& =\$ 256
\end{aligned}
$$

### 8.4 Like Terms

Occasionally terms look alike in an expression. These like-terms can and should be combined.
$\mathbf{2}+\mathbf{3}=\mathbf{5} . \mathbf{2}$ apples added to $\mathbf{3}$ apples results in $\mathbf{5}$ apples.
$2 x+3 x=(x+x)+(x+x+x)=5 x$
$2 x^{2}+3 x^{2}=\left(x^{2}+x^{2}\right)+\left(x^{2}+x^{2}+x^{2}\right)=5 x^{2}$
Don't confuse with $\left(2 x^{2}\right)\left(3 x^{2}\right)=(2)(3)(x \cdot x)(x \cdot x)=6 x^{4}$
Example 12:
Combine like terms: $\mathbf{2} \boldsymbol{x}^{\mathbf{3}}+\mathbf{5 x}+\mathbf{9}+\mathbf{6} \boldsymbol{x}^{\mathbf{3}}+\boldsymbol{x}-\mathbf{9}$

## Solution:

$$
\begin{aligned}
& 2 x^{3}+5 x+9+6 x^{3}+x-9 \\
= & \underline{2 x^{3}}+\underline{\underline{5 x}}+\underline{\underline{\underline{9}}}+\underline{6 x^{3}}+\underline{\underline{x}}-\underline{\underline{\underline{9}}} \text { mark each like-term with its own symbol } \\
= & \left(\underline{2 x^{3}}+\underline{6 x^{3}}\right)+(\underline{\underline{5 x}}+\underline{\underline{x}})+(\underline{\underline{\underline{9}}}-\underline{\underline{\underline{9}}})=8 x^{3}+6 x+0
\end{aligned}
$$

Example 13:
Combine like terms: $7 x^{3}+4\left[5(x+8)+6 x^{3}-x-2\right]$

## Solution:

$$
\begin{aligned}
& 7 x^{3}+4\left[5(x+8)+6 x^{3}-x-2\right] \\
= & 7 x^{3}+4\left[5(x+8)+6 x^{3}-x-2\right] \text { Remember PEMDAS? Focus on innermost group (parentheses). } \\
& \text { Addition is the only operation. We cannot add an unknown (value unknown) to a constant(a } \\
& \text { known non-varying number). } \\
= & 7 x^{3}+4\left[5 x+5(8)+6 x^{3}-x-2\right] \text { Use the distributive property of multiplication over addition } \\
& \quad \text { to remove the parentheses. Remember that } x \text { is a number. } \\
= & 7 x^{3}+4\left[\underline{5 x}+\underline{\underline{40}}+6 x^{3}-\underline{x}-\underline{2}\right] \text { Underline like-terms. } \\
= & \mathbf{7} x^{3}+4\left[(5 x-x)+(40-2)+6 x^{3}\right] \text { Associate like-terms. } \\
= & \mathbf{7} x^{3}+4\left[4 x+38+6 x^{3}\right] \text { Compute. } \\
= & \mathbf{7 x}+4\left[6 x^{3}+4 x+38\right] \text { Rewrite with exponents in decreasing order. } \\
= & \mathbf{7} x^{3}+4\left(6 x^{3}\right)+4(4 x)+4(38) \text { Distribute multiplication over addition. } \\
= & \mathbf{7 x}+\mathbf{2 4} x^{3}+\mathbf{1 6 x}+\mathbf{1 5 2} \text { Compute. } \\
= & \mathbf{3 1} x^{3}+\mathbf{1 6 x}+\mathbf{1 5 2}
\end{aligned}
$$

### 8.5 Exercises 8

1. Evaluate $P-(2 L+2 W)$ if $P=50, L=9$ and $W=4$.
2. Evaluate $\boldsymbol{S}-\left(\mathbf{2} \boldsymbol{x}^{2}+\mathbf{2 x y}\right)$ if $\boldsymbol{S}=\mathbf{1 0 0}, \boldsymbol{x}=-\mathbf{3}$ and $\boldsymbol{y}=\mathbf{5}$.
3. Evaluate $x-\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}$ if
$x=10, a=1, b=-4$ and $c=-21$.
4. Evaluate $\boldsymbol{D}-\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}}+\boldsymbol{v} \boldsymbol{t}+\boldsymbol{h}$ if
$D=200, t=3, v=20$ and $h=128$.
5. Evaluate $S-a^{2}+b^{2}$ if $\boldsymbol{S}=\mathbf{1 6 9}, \boldsymbol{a}=\mathbf{1 2}$ and $\boldsymbol{b}=\mathbf{- 5}$.
6. Evaluate $y-\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right)$ if
$x=12, x_{1}=9, x_{2}=6$,
$y=19, y_{1}=-7$ and $y_{2}=-10$.
7. A bus travels a certain number (say $\boldsymbol{x}$ ) of miles on Interstate 405. Then on highway $\mathbf{5}$ the bus travels 17 more miles than on Interstate 405. The bus now turns to highway 18 and covers three times as many miles than on highway 18. The trip ends on local streets with the number of miles one quarter the number on Interstate 405.
(a) Write an expression for the total mileage traveled.
(b) If $\mathbf{3 2}$ miles were traveled on Interstate 405, what was the total mileage traveled on local streets?
8. Candy studies 144 pages for a test. Diane studies 120 pages for the same test. Candy gets $\boldsymbol{c}$ problems done per page studied and Diane finishes $\boldsymbol{d}$ problems per page.
(a) Write an expression for the total number of problems Candy and Diane solve for the test. (Assume the number of problems on each page is the same.)
(b) Then evaluate that expression if Candy completes $\mathbf{3}$ problems per page and Diane succeeds in finishing 4 problems per page.
9. Simplify $7 x^{4}+11 x^{2}-4 x+9+3 x^{4}-11 x^{3}+4 x+9$
10. Combine like terms:

$$
9 x^{4}+7\left[3(x+1)-6 x^{4}-x+2\right]
$$

STOP!

1. Evaluate $\boldsymbol{P}-(2 L+2 W)$ if $\boldsymbol{P}=50, L=9$ and $W=4$.

## Solution:

$$
\begin{aligned}
P-(2 L+2 W) & =()-[2()+2()] \\
& =(50)-[2(9)+2(4)] \\
& =50-(18+8) \\
& =50-26=24
\end{aligned}
$$

2. Evaluate $S-\left(2 \boldsymbol{x}^{2}+\mathbf{2 x y}\right)$ if $\boldsymbol{S}=\mathbf{1 0 0}, \boldsymbol{x}=-\mathbf{3}$ and $\boldsymbol{y}=\mathbf{5}$.

## Solution:

$$
\begin{aligned}
S-\left(2 x^{2}+2 x y\right) & =()-\left[2()^{2}+2()()\right] \\
& =(100)-\left[2(-3)^{2}+2(-3)(5)\right] \\
& =100-[2(9)+(-6)(5)] \\
& =100-[18+(-30)] \\
& =100-(-12)=112
\end{aligned}
$$

3. Evaluate $x-\frac{-b+\sqrt{b^{2}-4 a c}}{2 a}$ if $x=10, a=1, b=-4$ and $c=-21$.

Solution:

$$
\begin{aligned}
x-\frac{-b+\sqrt{b^{2}-4 a c}}{2 a} & =()-\frac{-()+\sqrt{()^{2}-4()()}}{2()} \\
& =(10)-\frac{-(-4)+\sqrt{(-4)^{2}-4(1)(-21)}}{2(1)} \\
& =10-\frac{4+\sqrt{16-4(-21)}}{2} \\
& =10-\frac{4+\sqrt{16+84}}{2} \\
& =10-\frac{4+\sqrt{100}}{2} \\
& =10-\frac{4+10}{2} \\
& =10-\frac{14}{2} \\
& =10-7=3
\end{aligned}
$$

4. Evaluate $\boldsymbol{D}-\mathbf{1 6 t} \boldsymbol{t}^{2}+\boldsymbol{v} \boldsymbol{t}+\boldsymbol{h}$ if $\boldsymbol{D}=\mathbf{2 0 0}, \boldsymbol{t}=\mathbf{3}, \boldsymbol{v}=\mathbf{2 0}$ and $\boldsymbol{h}=\mathbf{1 2 8}$.

## Solution:

$$
\begin{aligned}
D-16 t^{2}+v t+h & =()-16()^{2}+()()+() \\
& =(200)-16(3)^{2}+(20)(3)+(128) \\
& =200-16(9)+60+128 \\
& =200-144+60+128 \\
& =56+60+128 \\
& =116+128 \\
& =244
\end{aligned}
$$

5. Evaluate $\boldsymbol{S}-\boldsymbol{a}^{2}+\boldsymbol{b}^{2}$ if $\boldsymbol{S}=\mathbf{1 6 9}, \boldsymbol{a}=\mathbf{1 2}$ and $\boldsymbol{b}=\mathbf{- 5}$.

Solution:

$$
\begin{aligned}
S-a^{2}+b^{2} & =()-()^{2}+()^{2} \\
& =(169)-(12)^{2}+(-5)^{2} \\
& =169-144+25 \\
& =25+25 \\
& =50
\end{aligned}
$$

6. Evaluate $y-\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right)$ if $x=12, x_{1}=9, x_{2}=6, y=19 y_{1}=-7$ and $y_{2}=-10$.

Solution:

$$
\begin{aligned}
y-\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right) & =()-\frac{()-()}{()-()}\left[(x)-\left(x_{1}\right)\right] \\
& =(19)-\frac{(-10)-(-7)}{(6)-(9)}[(12)-(9)] \\
& =19-\frac{-10+7}{6-9}(12-9) \\
& =19-\frac{-3}{-3}(3) \\
& =19-(1)(3) \\
& =16
\end{aligned}
$$

7. A bus travels a certain number (say $\boldsymbol{x}$ ) of miles on Interstate 405. Then on highway $\mathbf{5}$ the bus travels 17 more miles than on Interstate 405. The bus now turns to highway 18 and covers three times as many miles than on Interstate 405. The trip ends on local streets with the number of miles one quarter the number on Interstate 405.
(a) Write an expression for the total number of miles traveled.
(b) If $\mathbf{3 2}$ miles were traveled on Interstate 405, what was the total mileage traveled on local streets?

## Solution:

Miles traveled on Interstate $405 \quad \boldsymbol{x}$

Miles traveled on highway $5 \quad \boldsymbol{x}+\mathbf{1 7}$
Miles traveled on highway $18 \quad \mathbf{3 ( x )}$
Miles covered on local streets
(a) Total number of miles:
$x+(x+17)+3 x+\frac{x}{4}$.
(b) Evaluate if $\boldsymbol{x}=\mathbf{3 2}$

$$
\begin{aligned}
& x+(x+17)+3 x+\frac{x}{4} \\
= & ()+[(x)+17]+3()+\frac{()}{4} \\
= & (32)+[(32)+17]+3(32)+\frac{(32)}{4} \\
= & 32+49+96+8 \\
= & 81+96+8 \\
= & 177+8 \\
= & 185
\end{aligned}
$$

The number of miles traveled on local streets is
$\frac{185}{4}=46.25$ miles .
8. Candy studies 144 pages for a test. Diane studies 120 pages for the same test. Candy gets $\boldsymbol{c}$ problems done per page studied and Diane finishes $\boldsymbol{d}$ problems per page.
(a) Write an expression for the total number of problems Candy and Diane solve for the test.
(b) Then evaluate that expression if Candy completes $\mathbf{3}$ problems per page and Diane succeeds in finishing 4 problems per page.

## Solution:

(a) Candy and Diane complete $\mathbf{1 4 4} \boldsymbol{c}+\mathbf{1 2 0 d}$ problems for the test.
(b) Evaluating $144 c+\mathbf{1 2 0 d}$ leads to

$$
\begin{aligned}
144 c+120 d & =144()+120() \\
& =144(3)+120(4) \\
& =432+480 \\
& =912
\end{aligned}
$$

9. Simplify $7 x^{4}+11 x^{2}-4 x+9+3 x^{4}-11 x^{3}+4 x+9$

Solution:
$7 x^{4}+11 x^{2}-4 x+9+3 x^{4}-11 x^{3}+4 x+9$
$=\underline{7 x^{4}}+\underline{\underline{\underline{11 x^{2}}}}-\underbrace{4 x}+\underbrace{\underbrace{9}}+\underline{3 x^{4}}-\underline{\underline{11 x^{3}}}+\underbrace{4 x}+\underbrace{\underbrace{9}}$
$=\underline{7 x^{4}}+\underline{3 x^{4}}-\underline{\underline{11 x^{3}}}+\underline{\underline{\underline{11 x^{2}}}}-\underbrace{4 x}+\underbrace{4 x}+\underbrace{9}+\underbrace{9}$
$=\left(7 x^{4}+3 x^{4}\right)-11 x^{3}+11 x^{2}+(-4 x+4 x)+(9+9)$
$=10 x^{4}-11 x^{3}+11 x^{2}+18$
10. Combine like terms:
$9 x^{4}+7\left[3(x+1)-6 x^{4}-x+2\right]$
Solution:

$$
\begin{aligned}
& 9 x^{4}+7\left[3(x+1)-6 x^{4}-x+2\right] \\
= & 9 x^{4}+7\left[3 x+3-6 x^{4}-x+2\right] \\
= & 9 x^{4}+7[\underline{3 x}+\underline{\underline{3}}-\underline{\underline{\underline{6 x}}}-\underline{x}+\underline{2}] \\
= & 9 x^{4}+7\left[-\underline{\underline{6 x^{4}}+\underline{3 x}-\underline{x}+\underline{3}+\underline{2}}\right] \\
= & 9 x^{4}+7\left[-6 x^{4}+(3 x-x)+(3+2)\right] \\
= & 9 x^{4}+7\left[-6 x^{4}+2 x+5\right] \\
= & 9 x^{4}+7\left(-6 x^{4}\right)+7(2 x)+7(5) \\
= & \left(9 x^{4}-42 x^{4}\right)+14 x+35 \\
= & -33 x^{4}+14 x+35
\end{aligned}
$$

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## Chapter 9

## Solving Linear Equations by Addition/Subtraction

### 9.1 Youtube

http://www.youtube.com/playlist?list=PL59D38607B40A2E85\&feature=view_all
(C) H. Feiner 2011

### 9.2 Definition of an Equation

An equation is a mathematical statement that has two expressions, separated by an equal sign, which are of the same value. The two sides of the equation are called members.
$5+3=8$
$x+3=8$
are examples of equations.
Think of an equation as a balanced scale.


If a balance is in equilibrium, adding and/or subtracting the same number to/from both scales will keep the balance in equilibrium.

$3+5=8$


$$
2+3+5=2+8
$$

if $x+5=8$

then $2+x+5=2+8$

if $x+5=8$
then $\boldsymbol{x}+\mathbf{5 - 7}=\mathbf{8 - 7}$

Given an equation $x=a$
Add $b$ to both sides (members) $x+b=a+b$
Subtract $b$ from both sides $x-b=a-b$

### 9.3 Reversing the Operation of Addition/Subtraction

Being able to add or subtract the same number to or from both sides of an equation is straightforward. An addition can be used to cancel (void, reverse) the effect of a subtraction.

If $\boldsymbol{x}+4=9$
then
$x+4-4=9-4$
or
$x=5$.
In this example one can guess the solution $\boldsymbol{x}=\mathbf{5}$ by observation. A solution of an equation (usually) is a number that satisfies the equation, that is makes the two sides equal.

This cancellation by addition/subtraction is extremely useful in more complicated situations. Be careful later when you need to cancel (void, reverse) multiplication or division. These latter operations cannot be canceled by either addition or subtraction.

Example 1:
Is $\mathbf{- 3}$ a solution of $\mathbf{2 x}+\mathbf{6}=\mathbf{9}$ ?

## Solution:

$$
\begin{aligned}
2 x+6 & =12 \\
2()+6 & =12
\end{aligned} \text { Ready to receive the value for the unknown. }
$$

Example 2:
Is $\mathbf{3}$ a solution of $\mathbf{2 y}+\mathbf{6}=12$ ?

## Solution:

$$
\begin{aligned}
2 y+6 & =12 \\
2()+6 & =12
\end{aligned} \text { Ready to receive the value for the unknown. }
$$

## Example 3:

Is 1.3 a solution of $\mathbf{3}(\boldsymbol{x}-\mathbf{0 . 8})+\mathbf{2 . 5}=\mathbf{2 . 7}+\boldsymbol{x}$ ?

## Solution:

$$
\begin{array}{rll}
3(x-0.8)+2.5 & =2.7+\boldsymbol{x} & \\
3(()-0.8)+2.5 & =2.7+() & \\
\text { Ready to get the value for the unknown. } \\
3((1.3)-0.8)+2.5 & =2.7+(1.3) & \text { Introduce the value. } \\
\mathbf{3 ( 1 . 3 - 0 . 8 ) + 2 . 5}=2.7+1.3 & \text { Subtract. } \\
\mathbf{3 ( 0 . 5 ) + 2 . 5}=4.0 & \text { Multiply. } \\
1.5+2.5=4.0 & \text { Add. } \\
4.0 & =4.0 & \text { The left side is equal to the right side. } \\
& & 1.3 \text { is a solution of the equation. }
\end{array}
$$

In solving equations, it is a good idea to end up with all the unknowns on one side and the constants on the other side.

Example 4:
Is $\frac{1}{8}$ a solution of $3 x+5 x-4(1-x)=-20 x ?$
Solution:

$$
\begin{aligned}
& 3 x+5 x-4(1-x)=-20 x \\
& 3()+5()-4(1-())=-20() \quad \text { Ready to get the value for the unknown. } \\
& 3\left(\frac{1}{8}\right)+5\left(\frac{1}{8}\right)-4\left(1-\left(\frac{1}{8}\right)\right)=-20\left(\frac{1}{8}\right) \quad \text { Populate. } \\
& \frac{3}{8}+\frac{5}{8}-4\left(\frac{8}{8}-\frac{1}{8}\right)=\frac{-5}{2} \quad \text { Prepare fraction subtraction, reduce right side } \\
& \frac{3+5}{8}-4\left(\frac{7}{8}\right)=\frac{-5}{2} \quad \text { Subtraction. } \\
& \frac{8}{8}-\frac{7}{2}=\frac{-5}{2} \\
& \frac{2}{2}-\frac{7}{2}=\frac{-5}{2} \\
& \text { Reduce. } \\
&-\frac{5}{2}=\frac{-5}{2} \text { Reduce. }
\end{aligned}
$$

$\frac{1}{8}$ is a solution of the equation.

Example 5:
Solve $\boldsymbol{t}-\mathbf{6}=13$.

## Solution:

$$
\begin{aligned}
t-6 & =13 \\
t-6+6 & =13+6 \\
t & =19
\end{aligned}
$$

Example 6:
Solve $3 x-\frac{2}{9}=2 x+\frac{5}{12}$.

## Solution:

$$
\begin{aligned}
& 3 x-\frac{2}{9}=2 x+\frac{5}{12} \\
& -2 \boldsymbol{x} \quad-2 \boldsymbol{x} \quad \text { Subtract } 2 \boldsymbol{x} \text { from both sides. } \\
& x-\frac{2}{9}=\frac{5}{12} \\
& \begin{array}{lll}
\frac{\mathbf{2}}{\mathbf{9}} & \frac{\mathbf{2}}{\mathbf{9}} & \text { Add } \frac{\mathbf{2}}{\mathbf{9}} \text { to both sides. }
\end{array} \\
& x=\frac{5}{12}+\frac{2}{9} \\
& \boldsymbol{x}=\frac{\mathbf{5}}{\mathbf{1 2}}+\frac{\mathbf{2}}{\mathbf{9}} \quad \text { The LCD is } \mathbf{3 6} \\
& x=\frac{5 \cdot 3}{12 \cdot 3}+\frac{2 \cdot 4}{9 \cdot 4} \\
& x=\frac{15}{36}+\frac{8}{36} x=\frac{23}{36}
\end{aligned}
$$

Multiplying both sides of the original equation by the LCD is usually a better way of solving an equation with fractions. We'll come back to this problem.

Example 7:
Solve $5 x-9+4 x-2=8 x+11-22$.

## Solution:

| $5 x-9+4 x-2$ | $=8 x+11-22$ |  |  |
| ---: | :--- | ---: | :--- |
| $(5 x+4 x)-(9+2)$ | $=8 x-(22-11)$ |  | Combine like-terms |
| $9 x-11$ | $=8 x-11$ |  | Compute. |
| 11 |  | 11 |  |
| $9 x$ | $=8 x$ |  | Add 11 to both sides. |
| $-8 x$ |  | $-8 x$ |  |
| Compute. |  |  |  |
|  | $=0$ |  | Subtract $8 x$ from both sides. |

Example 8:
President Hayes died at a certain age. President Adams died at age 20 years more than Hayes. President Theodore Roosevelt died at an age $\mathbf{1 0}$ years less than Hayes. The sum of all three ages equals $\mathbf{9 0}$ more than the sum of Hayes and Roosevelt.

How old was President Hayes when he died?
Hint: Let $\boldsymbol{x}$ be the age at which President Hayes died.

## Solution:

President Adams' age at death: $\boldsymbol{x}+\mathbf{2 0}$
President Hayes' age at death: $\boldsymbol{x}$
President Roosevelt's age at death: $\boldsymbol{x} \boldsymbol{- 1 0}$
The sum of all three ages: $\boldsymbol{x}+\mathbf{2 0}+\boldsymbol{x}+\boldsymbol{x}-\mathbf{1 0}=\mathbf{3 x}+\mathbf{1 0}$.
90 more than the sum of Hayes and Roosevelt: $\boldsymbol{x}+\boldsymbol{x}-\mathbf{1 0}+\mathbf{9 0}=\mathbf{2 x}+\mathbf{8 0}$.
Equation:

| $3 x+10$ | $=$ | $2 x+80$ |
| ---: | :---: | :---: |
| -10 |  | -10 |
| $3 x$ | $=$ | $2 x+70$ |
| $-2 x$ |  | $-2 x$ |
| $x$ | $=70$ |  |

President Hayes was 70 years old when he died.

### 9.4 Exercises 9

1. Is -5 a solution of $\mathbf{6 x - 8}=\mathbf{3 8}$ ?
2. Is $\mathbf{1}$ a solution of $\mathbf{2 y}-14=12(y-2)$ ?
3. Is 2.1 a solution of $5(x-0.6)-5.2=2.9-x$ ?
4. Is $\frac{1}{6}$ a solution of $2 x+4 x-6(1-x)=-24 x$ ?
5. Solve $\boldsymbol{p}-\mathbf{7}=\mathbf{2 2}$.
6. Solve $4 y-\frac{3}{10}=3 y+\frac{4}{15}$.
7. Solve $5(x-9)+4(x-2)=8(x+11)$.
8. The gestation for an elk is a certain number.

The gestation for a chimpanzee is $\mathbf{2 0}$ days less.
The gestation for a horse is $\mathbf{8 0}$ days more than an elk.
The sum of all three gestations equals $\mathbf{1 5 0}$ less than twice the sum of the gestations of a chimpanzee and an elk.
What is the gestation of an elk?
Hint: Let $\boldsymbol{x}$ be the gestation of an elk.

## STOP!

1. Is -5 a solution of $\mathbf{6 x - 8}=\mathbf{3 8}$ ?

## Solution:

$$
\begin{aligned}
& 6 x-8=38 \\
& 6()-8=38 \\
& 6(-5)-8=38 \\
&-30-8=38 \\
&-38=38 \text { Subpalate the right side is not equal to the left side. } \\
&-38 \\
& x=-5 \text { is not a solution. }
\end{aligned}
$$

2. Is $\mathbf{1}$ a solution of $\mathbf{2 y}-14=12(y-2)$ ?

## Solution:

$$
\begin{aligned}
2 y-14 & =12(y-2) & & \\
2()-14 & =12(()-2) & & \text { Prepare to receive a value. } \\
2(1)-14 & =12((1)-2) & & \text { Populate. } \\
2-14 & =12(-1) & & \text { Compute. } \\
-12 & =-12 & & \text { The right side is equal to the left side. }
\end{aligned}
$$

$\boldsymbol{y}=\mathbf{1}$ is a solution of the equation.
3. Is 2.1 a solution of $\mathbf{5}(\boldsymbol{x}-\mathbf{0 . 6})-\mathbf{5 . 2}=\mathbf{2 . 9}-\boldsymbol{x}$ ?

## Solution:

$$
\begin{array}{rll}
5(\boldsymbol{x}-\mathbf{0 . 6})-5.2 & =2.9-x & \\
5(()-0.6)-5.2 & =2.9-() & \\
5((2.1)-0.6)-5.2 & =2.9-(\mathbf{2 . 1}) & \text { Popare to receive a value. } \\
\mathbf{5 ( 2 . 1 - 0 . 6 ) - 5 . 2} & =2.9-\mathbf{2 . 1} & \text { Compute. } \\
5(1.5)-5.2 & =0.8 & \text { Compute. } \\
7.5-5.2 & =0.8 & \text { Compute. } \\
2.3 & =0.8 & \text { The right side is not equal to the left side. }
\end{array}
$$

$\boldsymbol{x}=2.1$ is a not solution of the equation.
4. Is $\frac{1}{6}$ a solution of $2 x+4 x-6(1-x)=-24 x$ ?

## Solution:

$$
\begin{aligned}
2 x+4 x-6(1-x) & =-24 x \\
2()+4()-6(1-()) & =() \\
2\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)-6\left(1-\left(\frac{1}{6}\right)\right) & =24\left(\frac{1}{6}\right) \\
\frac{2}{6}+\frac{4}{6}-6\left(\frac{6}{6}-\frac{1}{6}\right) & =-4 \\
\frac{2+4}{6}-\frac{6(6-1)}{6} & =-4 \\
1-5 & =-4
\end{aligned}
$$

$$
-4=-4 \quad \text { The left side equals the right side. }
$$

Yes, $\frac{\mathbf{1}}{\mathbf{6}}$ is a solution.
5. Solve $\boldsymbol{p}-\mathbf{7}=\mathbf{2 2}$.

## Solution:

$$
\begin{array}{r}
p-7=22 \\
7
\end{array}
$$

6. Solve $4 y-\frac{3}{10}=3 y+\frac{4}{15}$.

## Solution:

$$
\begin{aligned}
4 y-\frac{3}{10} & =3 y+\frac{4}{15} \\
-3 y & -3 y \\
y-\frac{3}{10}= & \frac{4}{15} \\
\frac{3}{10} & \frac{3}{10} \\
y & =\frac{4}{15}+\frac{3}{10} \\
y & =\frac{4 \cdot 2}{15 \cdot 2}+\frac{3 \cdot 3}{10 \cdot 3} \\
y & =\frac{8}{30}+\frac{9}{30} \\
y & =\frac{8+9}{30} y=\frac{17}{30}
\end{aligned}
$$

7. Solve $5(x-9)+4(x-2)=8(x+11)$.

Solution:

| $5(x-9)+4(x-2)$ | $=8(x+11)$ |
| ---: | :--- |
| $5 x-5(9)+4 x-4(2)$ | $=8 x+8(11)$ |
| $5 x-45+4 x-8$ | $=8 x+88$ |
| $(5 x+4 x)-(45+8)$ | $=8 x+88$ |
| $9 x-53$ | $=8 x+88$ |
| $-8 x$ |  |
| $x-53$ | $=88$ |
| +53 |  |
| $x$ | $=88+53$ |
| $x$ | $=141$ |

8. The gestation for an elk is a certain number.

The gestation for a chimpanzee is $\mathbf{2 0}$ days less.
The gestation for a horse is $\mathbf{8 0}$ days more than an elk.
The sum of all three gestations equals $\mathbf{1 5 0}$ less than twice the sum of the gestations of a chimpanzee and an elk.
What is the gestation of an elk?
Hint: Let $\boldsymbol{x}$ be the gestation of an elk.

## Solution:

Gestation time of an elk: $\boldsymbol{x}$.
Gestation time of a chimpanzee: $\boldsymbol{x}-\mathbf{2 0}$.
Gestation time of a horse: $\boldsymbol{x}+\mathbf{8 0}$.
Sum of all three gestations: $x+x-20+x+80=3 x+60$
Sum of gestations of chimpanzee \& elk: $\boldsymbol{x}-\mathbf{2 0}+\boldsymbol{x}=\mathbf{2 x} \mathbf{- 2 0}$.
Twice the last sum: $2(2 x-20)=4 x-40$.
Equation:

| $3 x+60$ | $=$ | $4 x-40-150$ |
| ---: | :--- | :---: |
| $3 x+60$ | $=$ | $4 x-190$ |
| $-3 x$ |  | $-3 x$ |
| 60 | $=$ | $x-190$ |
| 190 | 190 |  |
| 250 | $=x$ |  |

The gestation of an elk is $\mathbf{2 5 0}$ days.

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## Chapter 10

## Solving Linear Equations by Multiplication/Division

### 10.1 Youtube

http://www.youtube.com/playlist?list=PL3A2DC3F523D85FC2\&feature=view_all

### 10.2 Reversing Multiplication/Division

Suppose $2+5$ pounds of grain are loaded on the left scale of a balance. At the same time $\mathbf{7}$ pounds are put on the right side.

If the number of pounds on the left is doubled $2(2+5)$ and simultaneously the number on the right side is doubled $2(7)$, the balance will remain in equilibrium.

Similarly we can divide the left side by 7 if we divide the right side by 7 .
$\frac{2+5}{7}=\frac{7}{7}$

# Given an equation $a x=b$ <br> Multiply both sides by $c$ to get $a c x=b c$ <br> Divide both sides by $a \neq 0$ to get $\frac{a x}{a}=x=\frac{b}{a}$ 

## The Grand Canyon

separates addition/subtraction from multiplication/division.

> Given an equation $x=a$
> Add $b$ to both sides (members) $x+b=a+b$
> Subtract $b$ from both sides $x-b=a-b$

Being able to multiply or divide by the same number on both sides of an equation is straightforward. A multiplication can be used to cancel (void, reverse) the effect of a division and vice-versa.

If $3 x=15$
then

$$
\frac{3 x}{3}=\frac{15}{3} \text { or } x=5
$$

Could we have subtracted $\mathbf{3}$ from both sides of the original equation $\mathbf{3 x}=\mathbf{1 5}$ ?
Yes, $\mathbf{3 x - 3}=\mathbf{1 5}-\mathbf{3}$ is correct but counter productive. We need to get to a stage where only unknowns are on one side and constants on the other side of an equation.

Note that addition and subtraction undo each other. Multiplication and division undo each other also. Do not try to undo a multiplication by addition or subtraction.

If a member of an equation consists of more than one term. multiply/divide all the terms of the member by the number multiplied/divided on both sides.

If $\mathbf{3 x}+\mathbf{6}=\mathbf{2 1}$ (you should first subtract $\mathbf{6}$ from both sides, then divide by $\mathbf{3}$ )

$$
\begin{aligned}
\frac{3 x+6}{3} & =\frac{21}{3} \\
\frac{3 x}{3}+\frac{6}{3} & =7 \\
x+2 & =7 \\
x & =5
\end{aligned}
$$

but if you insist on dividing then $\frac{\mathbf{3 x}}{\mathbf{3}}+\frac{6}{3}=7$

Example 1:
Solve $\frac{4}{5} x=-8$.

## Solution:

$$
\begin{aligned}
& \frac{\mathbf{4}}{\mathbf{5}} \boldsymbol{x}=-\mathbf{8} \\
& \frac{\mathbf{5}}{\mathbf{4}} \cdot \frac{\mathbf{4}}{\mathbf{5}} \boldsymbol{x}=-\mathbf{8} \cdot \frac{\mathbf{5}}{4} \text { division by a fraction is equivalent to } \\
& \text { multiplication by the reciprocal of the fraction. } \\
& \boldsymbol{x}=-\mathbf{2} \cdot \frac{\mathbf{5}}{\mathbf{1}} \text { Reduce. } \\
& \boldsymbol{x}=-\mathbf{1 0}
\end{aligned}
$$

Example 2:
Solve $2(y+6)=5(y-12) ?$

## Solution:

$$
\begin{array}{rlrl}
2(y+6) & =5(y-12) & & \begin{array}{l}
\text { Distributive multiplication over addition } \\
\text { and subtraction to "free" the variables. }
\end{array} \\
2 y+2(6) & =5 y-5(12) \\
2 y+12 & =5 y-60 \quad \text { Multiply. } \\
2 y+12-12 & =5 y-60-12 \quad \text { Subtract } 12 \text { from both sides. } \\
2 y & =5 y-72 \\
-3 y & =-72 \\
\frac{-3}{-3} y & =\frac{-72}{-3} & \text { Subtract } 5 y \text { from both sides. } \\
y & =24 & \text { Divide both sides by }-3
\end{array}
$$

Example 3:
Solve $3(x-0.8)+2.5 x=2.7+5 x$.

## Solution:

$$
\begin{aligned}
3(x-0.8)+2.5 x & =2.7+5 x \\
3 x-3(0.8)+2.5 x & =2.7+5 x \quad \text { Distribute multip. over subtraction. } \\
3 x-2.4+2.5 x & =2.7+5 x \quad \text { Multiply. } \\
5.5 x-2.4 & =2.7+5 x \quad \text { Combine like-terms. } \\
5.5 x-5 x-2.4 & =2.7+5 x-5 x \quad \text { Subtract } 5 x \text { from both sides. } \\
0.5 x-2.4 & =2.7 \quad \text { Subtract. } \\
0.5 x-2.4+2.4 & =2.7+2.4 \quad \text { Add } 2.4 \text { to both sides. } \\
0.5 x & =5.1 \quad \text { Add. Variables are segregated on } \\
\frac{0.5}{0.5} x & =\frac{5.1}{0.5} \quad \text { the left, constants on the right side. } \\
x & =2(5.1) \text { This is is equivalent to multiplying by } 2 . \\
x & =10.2
\end{aligned}
$$

Example 4:
Solve $\frac{3}{10} x+\frac{7}{15} x-\frac{11}{30}(1-x)=\frac{1}{5} x$.
Solution:

$$
\begin{aligned}
\frac{3}{10} x+\frac{7}{15} x-\frac{11}{30}(1-x) & =\frac{1}{5} x \\
\frac{3 \cdot 30}{10} x+\frac{7 \cdot 30}{15} x-\frac{11 \cdot 30}{30}(1-x) & =\frac{1 \cdot 30}{5} x \text { Multiply by LCD }=30 \\
\frac{3 \cdot 3}{1} x+\frac{7 \cdot 2}{1} x-\frac{11 \cdot 1}{1}(1-x) & =\frac{1 \cdot 6}{1} x \text { Reduce. } \\
9 x+14 x-11(1-x) & =6 x \quad \text { Simplify. } \\
9 x+14 x-11-(-11) x & =6 x \text { Distribute multiplication over subtraction. } \\
9 x+14 x-11+11 x & =6 x \quad \text { Rewrite subtraction. } \\
34 x-11 & =6 x \quad \text { Compute. } \\
34 x-34 x-11 & =6 x-34 x \text { Subtract }-34 x \text { from } 2 \text { sides. } \\
-11 & =-28 x \text { Compute. } \\
-28 x & =-11 \\
\frac{-28}{-28} x & =\frac{-11}{-28} \text { Divide both sides by }-28 \\
x & =\frac{11}{28}
\end{aligned}
$$

Example 5:
Solve $2(t-7)+t=3(t+1)$.

## Solution:

$$
\begin{aligned}
2(t-7)+t & =3(t+1) & & \begin{array}{l}
\text { Distribute multiplication over } \\
\text { addition and subtraction. }
\end{array} \\
3 t-14 & =3 t+3 & & \text { Combine like-terms. } \\
3 t-3 t-14 & =3 t-3 t+3 & & \text { Subtract } 3 t \text { from both sides. } \\
-\mathbf{1 4} & =3 & &
\end{aligned}
$$

This is false. No matter what you replace $\boldsymbol{t}$ by in the original equation, the false statement cannot be remedied.

There are no $\boldsymbol{x}$ s in the final step.
The equation has no solution.
Example 6:
Solve $3 x-\frac{2}{9}=4 x-\frac{1}{18}+\frac{1}{6}-x-\frac{1}{3}$.

## Solution:

$$
\begin{aligned}
& 3 x-\frac{2}{9}=4 x-\frac{1}{18}+\frac{1}{6}-x-\frac{1}{3} \text { Multiply } 2 \text { sides by LCD }=18 \\
& 18 \cdot 3 x-\frac{2 \cdot 18}{9}=18 \cdot 4 x-\frac{1 \cdot 18}{18}+\frac{1 \cdot 18}{6}-18 \cdot x-\frac{1 \cdot 18}{3} \text { Reduce. } \\
& 54 x-\frac{2 \cdot 2}{1}=72 x-\frac{1 \cdot 1}{1}+\frac{1 \cdot 3}{1}-18 x-\frac{6}{1} \text { Simplify } . \\
& 54 x-4=72 x-1+3-18 x-6 \text { Simplify. } \\
& 54 x-4=54 x+2-6 \text { Combine like-terms. } \\
& 54 x-4=54 x-4 \text { Subtract. } \\
& 54 x-54 x-4=54 x-54 x-4 \text { Subtract } 54 x \text { from both sides. } \\
& -4=\mathbf{- 4} \text { Add } 4 \text { to both sides. } \\
& \mathbf{0}=\mathbf{0} \text { This is true. No matter what you replace } \boldsymbol{x} \text { by in } \\
& \text { the original equation, the true statement cannot be falsified. } \\
& \text { There are no } \boldsymbol{x} \text { s in the final step. }
\end{aligned}
$$

There are infinitely many solutions in the original equation.
Every real number is a solution, even $\boldsymbol{\pi}$ with its infinite number of digits.
Multiplying both sides of the original equation by the LCD is usually a better way of solving an equation with fractions.

Example 7:

Solve $3 x-\frac{2}{9}=2 x+\frac{5}{12}$.

## Solution:

$$
\begin{array}{rlrl}
3 x-\frac{2}{9} & =2 x+\frac{5}{12} & \text { Multiply } 2 \text { sides by LCD }=36 . \\
36 \cdot 3 x-\frac{2 \cdot 36}{9} & =2 \cdot 36 x+\frac{5 \cdot 36}{12} & \text { Reduce. } \\
108 x-\frac{2 \cdot 4}{1} & =72 x+\frac{5 \cdot 3}{1} & & \text { Simplify } \\
108 x-8 & =72 x+15 & & \text { Simplify. } \\
108 x-8+8 & =72 x+15+8 & & \text { Add } 8 \text { to both sides. } \\
108 x & =72 x+23 & & \text { Add. } \\
108 x-72 x & =72 x-72 x+23 & & \text { Subtract } \mathbf{7 2 x} \text { from both sides. } \\
36 x & =23 & & \text { Simplify. } \\
36 & & & \text { Divide both sides by } \mathbf{3 6} . \\
x & =\frac{23}{36} &
\end{array}
$$

Example 8:
At midnight a liter $(1,000 \mathrm{~mL})$ of sodium chloride solution is injected in a patient at the rate of $\frac{\mathbf{7 5} \mathrm{mL}}{\text { hour }}$. 700 mL are already in the patient's system. How much longer will it take the whole liter to be consumed?

## Solution:

Let $\boldsymbol{x}$ be the time (in minutes) needed to finish the sodium chloride solution.
The portion of the solution injected in $\boldsymbol{x}$ minutes is
$\frac{\boldsymbol{x} \text { minutes }}{\mathbf{1}} \cdot \frac{1 \text { hour }}{\mathbf{6 0} \text { minutes }} \cdot \frac{\mathbf{7 5} \mathrm{mL}}{\text { hour }}=\frac{\boldsymbol{x}}{\mathbf{1}} \cdot \frac{\mathbf{1}}{4 \cdot \mathbf{1 5}} \cdot \frac{\boldsymbol{5} \cdot \mathbf{1 5} \mathrm{~mL}}{\mathbf{1}}=\frac{\mathbf{5}}{4} \boldsymbol{x} \mathrm{~mL}$.
Units are so useful (factor out $\frac{\mathbf{1 5}}{\mathbf{1 5}}=\mathbf{1}$, "cancel" units). The patient is still getting $\mathbf{1 , 0 0 0}-\mathbf{7 0 0}=\mathbf{3 0 0}$ mL .

Equation:

$$
\begin{aligned}
\frac{5 x \mathrm{~mL}}{4} & =300 \mathrm{~mL} \\
\frac{4}{5} \cdot \frac{5 x}{4} & =\frac{4}{5} \cdot 300 \\
x & =\frac{4}{1} \cdot 60 \\
x & =4 \cdot 60 \text { minutes }=4 \text { hours. The patient will have to be patient for four more hours. }
\end{aligned}
$$

Note: This problem can be solved more quickly by straightforward reasoning (keep units). The purpose here was to demonstrate how algebra can be used (in more complicated problems.)

Example 9:
Milan, Italy, produced a certain number of micrograms of particulate matter per cubic meter (mcgpcm) in 2004.

Dehli, India, generated five times that number.
Caracas, Venezuela, had 20 mcgpem less than Milan.
The sum of the numbers of particulates for all three cities equals $\mathbf{1 0}$ less than five times the sum of the number of particulates for Milan and Caracas.
Find the number of particulates for Milan.

## Solution:

Let $\boldsymbol{x}$ be the number of particulates in Milan.
Dehli generates five times that number: $\mathbf{5 x}$
Caracas had $\mathbf{2 0}$ mcgpem less than Milan: $\boldsymbol{x}-\mathbf{2 0}$
The sum of the numbers of particulates for all three cities is $\boldsymbol{x}+\mathbf{5 x + x} \mathbf{x} \mathbf{2 0}=\mathbf{7 x} \boldsymbol{x} \mathbf{2 0}$.
The sum of the numbers of particulates for Milan and Caracas is $\boldsymbol{x}+\boldsymbol{x}-\mathbf{2 0}=\mathbf{2 x}-\mathbf{2 0}$.
10 less than five times that sum is

$$
5(2 x-20)-10=10 x-100-10=10 x-110
$$

Equation:

| $\begin{array}{r} 7 x-20 \\ 20 \end{array}=$ | $\begin{array}{r} 10 x-110 \\ 20 \end{array}$ |
| :---: | :---: |
| $\begin{array}{r} 7 x \\ -10 x \end{array}=$ | $\begin{array}{r} 10 x-90 \\ -10 x \end{array}$ |
| $-3 x=$ | -90 |
| $\frac{-3}{-3} x$ | $\frac{-90}{-3}$ |
| $x=$ | 30 |

Milan had 30 mcgpem of particulates in 2004.
Example 10:
A telephone company used to charge a basic fee of $\mathbf{\$ 2 0}$ per month in addition to $\mathbf{1 0}$ cents per minute of use. Jimmy's bill for July was $\mathbf{\$ 5 0}$. For how many minutes did Jimmy use the phone in July?

## Solution:

Let $\boldsymbol{x}$ be the number of minutes used during July.
The cost is: basic fee $+\frac{\boldsymbol{x} \text { calls }}{\boldsymbol{1}} \cdot \frac{\mathbf{0 . 1} \text { dollars }}{\text { call }}$
Equation:

$$
\begin{aligned}
20+0.1 x & =50 \\
20-20+0.1 x & =50-20 \\
0.1 x & =30 \\
\frac{0.1}{0.1} x & =\frac{30}{0.1} \\
x & =\frac{300}{1}
\end{aligned}
$$

Jimmy spent $\mathbf{3 0 0}$ minutes ( $\mathbf{5}$ hours) on the phone in July.

### 10.3 Exercises 10

1. Solve $\frac{7}{5} x=-35$.
2. Solve $5(y-7)=2(y+13)$ ?
3. Solve $6(x-0.9)+2.7 x=2.8+0.5 x$.
4. Solve $\frac{7}{20} x-\frac{4}{25} x-\frac{13}{50}(2-x)=\frac{1}{5} x-\frac{1}{100} x$.
5. Solve $5(t-2)+2 t=7(t+3)$.
6. Solve $10 y-\frac{3}{8}=9 y-\frac{5}{24}-\frac{1}{4}+y+\frac{1}{12}$.
7. Solve $3 y+\frac{2}{7}=5 y+\frac{5}{21}$.
8. At midnight a liter $(\mathbf{1}, \mathbf{0 0 0} \mathrm{mL})$ of sodium chloride solution is injected in a patient at the rate of 50 mL . consumed?
9. According to the World Almanac, 2010 edition, National league statistics for 2009, the Arizona

Diamodbackers batter's averages at bat were as follows:
Gerardo Parra averaged a certain number at bat.
Justin Upton's average was $\mathbf{0 . 0 1}$ more.
MaxReynold's average was 0.3 less than Parra's average.
The sum of the averages for all three batters was $\mathbf{0 . 3 3}$ less than twice the average for Upton and Parra.
Find the average for Gerardo Parra.
10. A car company charges a basic fee of $\mathbf{\$ 2 0}$ per day in addition to $\mathbf{0 . 0 5}$ dollars per mile. Jimmy's budget for driving is $\mathbf{\$ 4 5}$. What is the maximum number of miles he can drive?

## STOP!

1. Solve $\frac{7}{5} \boldsymbol{x}=-\mathbf{3 5}$.

## Solution:

$$
\begin{aligned}
\frac{7}{5} x & =-35 \\
\frac{5}{7} \cdot \frac{7}{5} x & =-(35) \frac{5}{7} \\
x & =-(5) \frac{5}{1} \\
x & =-25
\end{aligned}
$$

2. Solve $5(y-7)=2(y+13)$ ?

## Solution:

$$
\begin{aligned}
5(y-7) & =2(y+13) \\
5 y-5(7) & =2 y+2(13) \\
5 y-35 & =2 y+26 \\
5 y-35+35 & =2 y+26+35 \\
5 y & =2 y+61 \\
5 y-2 y & =2 y-2 y+61 \\
3 y & =61 \\
\frac{3}{3} y & =\frac{61}{3} \\
y & =\frac{61}{3}
\end{aligned}
$$

3. Solve $6(x-0.9)+2.7 x=2.8+0.5 x$.

Solution:

$$
\begin{aligned}
6(x-0.9)+2.7 x & =2.8+0.5 x \\
6 x-6(0.9)+2.7 x & =2.8+0.5 x \\
6 x-5.4+2.7 x & =2.8+0.5 x \\
8.7 x-5.4 & =2.8+0.5 x \\
8.7 x-5.4+5.4 & =2.8+5.4+0.5 x \\
8.7 x & =8.2+0.5 x \\
8.7 x-0.5 x & =8.2+0.5 x-0.5 x \\
8.2 x & =8.2 \\
\frac{8.2}{8.2} x & =\frac{8.2}{8.2} \\
x & =1
\end{aligned}
$$

4. Solve $\frac{7}{20} x-\frac{4}{25} x-\frac{13}{50}(2-x)=\frac{1}{5} x-\frac{1}{100} x$.

Solution:

$$
\begin{aligned}
\frac{7}{20} x-\frac{4}{25} x-\frac{13}{50}(2-x) & =\frac{1}{5} x-\frac{1}{100} x \\
\frac{7 \cdot 100}{20} x-\frac{4 \cdot 100}{25} x-\frac{13 \cdot 100}{50}(2-x) & =\frac{1 \cdot 100}{5} x-\frac{1 \cdot 100}{100} x \\
\frac{7 \cdot 5}{1} x-\frac{4 \cdot 4}{1} x-\frac{13 \cdot 2}{1}(2-x) & =\frac{1 \cdot 20}{1} x-\frac{1}{1} x \\
35 x-16 x-26(2-x) & =20 x-x \\
35 x-16 x-52+26 x & =20 x-x \\
35 x+10 x-52 & =20 x-x \\
45 x-52 & =19 x \\
45 x-45 x-52 & =19 x-45 x \\
-52 & =-26 x \\
\frac{-52}{-26} & =\frac{-26}{-26} x \Rightarrow x=2
\end{aligned}
$$

5. Solve $5(t-2)+2 t=7(t+3)$.

## Solution:

$$
\begin{aligned}
5(t-2)+2 t & =7(t+3) \\
5 t-5(2)+2 t & =7 t+7(3) \\
5 t-10+2 t & =7 t+21 \\
7 t-10 & =7 t+21 \\
7 t-7 t-10 & =7 t-7 t+21 \\
-10 & =21
\end{aligned}
$$

False statement.
Can't be corrected.
The original equation has no solution.
6. Solve $10 y-\frac{3}{8}=9 y-\frac{5}{24}-\frac{1}{4}+y+\frac{1}{12}$.

## Solution:

$$
\begin{aligned}
& 10 y-\frac{3}{8}=9 y-\frac{5}{24}-\frac{1}{4}+y+\frac{1}{12} \\
& 10 \cdot 24 y-\frac{3 \cdot 24}{8}=9 \cdot 24 y-\frac{5 \cdot 24}{24}-\frac{1 \cdot 24}{4}+24 \cdot y+\frac{1 \cdot 24}{12} \\
& 240 y-\frac{3 \cdot 3}{1}=216 y-\frac{5 \cdot 1}{1}-\frac{1 \cdot 6}{1}+24 y+\frac{1 \cdot 2}{1} \\
& 240 y-9=216 y-5-6+24 y+2 \\
& 240 y-9=240 y-9 \\
& 240 y-240 y-9=240 y-240 y-9 \\
&-9=-9 \\
& 0=0 \text { True statement. Can't be falsified by } \\
& \text { using any value of } \boldsymbol{x} .
\end{aligned}
$$

The original equation has infinitely many solutions. Any real number is a solution.
7. Solve $3 y+\frac{2}{7}=5 y+\frac{5}{21}$.

## Solution:

$$
\begin{aligned}
3 y+\frac{2}{7} & =5 y+\frac{5}{21} \\
3 \cdot 21 y+\frac{2 \cdot 21}{7} & =5 \cdot 21 y+\frac{5 \cdot 21}{21} \\
63 y+\frac{2 \cdot 3}{1} & =105 y+\frac{5 \cdot 1}{1} \\
63 y+6 & =105 y+5 \\
63 y-63 y+6 & =105 y-63 y+5 \\
6 & =42 y+5 \\
6-5 & =42 y+5-5 \\
1 & =42 y \\
42 y & =1 \\
\frac{42}{42} y & =\frac{1}{42} \\
y & =\frac{1}{42}
\end{aligned}
$$

8. At midnight a liter $(\mathbf{1}, \mathbf{0 0 0} \mathrm{mL})$ of sodium chloride solution is injected in a patient at the rate of $\frac{\mathbf{5 0 ~ m L}}{\text { hour }} .400 \mathrm{~mL}$ are already in the patient's system. How much longer will it take the whole liter be consumed?

## Solution:

Let $\boldsymbol{x}$ be the time (in hours) needed to The portion of the solution injected in $\boldsymbol{x}$ hours is
$\frac{\boldsymbol{x} \text { hours }}{1} \cdot \frac{\mathbf{5 0 ~ m L}}{\text { hour }}=\mathbf{5 0 x} \mathrm{mL}$.
The units are so useful.
The patient is still getting $\mathbf{1 , 0 0 0}-\mathbf{4 0 0}=\mathbf{6 0 0} \mathrm{mL}$.
Equation:

$$
\begin{aligned}
50 x \mathrm{~mL} & =600 \mathrm{~mL} \\
\frac{50}{50} x & =\frac{600}{50} \\
x & =12
\end{aligned}
$$

The patient will have to be patient for 12 more hours.
Warning: The temptation is to solve this problem by inspection. You will not be able to find the answer by inspection on a test, because the numbers will not be this obvious. Learn to follow algebraic steps.
9. According to the World Almanac, 2010 edition, National league statistics for 2009, the Arizona Diamodbackers batters averages at bat were as follows:
Gerardo Parra averaged a certain number at bat.
Justin Upton's average was $\mathbf{0 . 0 1}$ more.
MaxReynold's average was $\mathbf{0 . 3}$ less than Parra's average.
The sum of the averages for all three batters was $\mathbf{0 . 3 3}$ less than twice the average for Upton and Parra.
Find the average for Gerardo Parra.

## Solution:

Let $\boldsymbol{x}$ be the average for Parra.
Upton's average was $\mathbf{0 . 0 1}$ more: $\boldsymbol{x}+\mathbf{0 . 0 1}$
Reynold's was $\mathbf{0 . 3}$ less than Parra's: $\boldsymbol{x} \mathbf{- 0 . 3}$
The sum of the averages for all three batters was
$x+x+0.01+x-0.3=3 x-0.29$.
The sum of the averages for Upton and Parra was
$x+x+0.01=2 x+0.01$.
The sum of the averages for all three batters $\mathbf{3 x}-\mathbf{0 . 2 9}$ was $\mathbf{0 . 3 3}$ less than twice the average for Upton and Parra $2 \boldsymbol{x}+\mathbf{0 . 0 1}$.
Equation:

$$
\begin{aligned}
3 x-0.29 & =2(2 x+0.01)-0.33 \\
3 x-0.29 & =4 x+0.02-0.33 \\
3 x-0.29 & =4 x-0.31 \\
3 x-3 x-0.29 & =4 x-3 x-0.31 \\
-0.29 & =x-0.31 \\
-0.29+0.31 & =x-0.31+0.31 \\
0.02 & =x
\end{aligned}
$$

Gerardo Parra's average at bat was $\mathbf{0 . 0 2 0}$.
10. A car company charges a basic fee of $\mathbf{\$ 2 0}$ per day in addition to $\mathbf{0 . 0 5}$ dollars per mile. Jimmy's budget for driving is $\mathbf{\$ 4 5}$. What is the maximum number of miles he can drive?

## Solution:

Let $\boldsymbol{x}$ be the maximum number of miles to be driven.
The cost is: basic fee $+\frac{\boldsymbol{x} \text { miles }}{\boldsymbol{1}} \cdot \frac{\mathbf{0 . 0 5} \text { dollars }}{\text { mile }}$
Equation:

$$
\begin{aligned}
20+0.05 x & =45 \\
20-20+0.05 x & =45-20 \\
0.05 x & =25 \\
\frac{0.05}{0.05} x & =\frac{25}{0.05} \\
x & =\frac{2500}{5}=500
\end{aligned}
$$

Jimmy can drive a maximum of $\mathbf{5 0 0}$ miles.

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## Chapter 11

## Solving Linear Equations. Integer Problems

### 11.1 Youtube

http://www.youtube.com/playlist?list=PLB5CEAC4D594DD31A\&feature=view_all

### 11.2 Basic Terminology

Consecutive Integers: 7, 8, $\mathbf{9}$ are consecutive integers. $\boldsymbol{x}, \boldsymbol{x}+\mathbf{1}, \boldsymbol{x}+\mathbf{2}$ are consecutive integers.
Consecutive Even Integers: 8, 10, $\mathbf{1 2}$ are consecutive even integers. $\boldsymbol{x}, \boldsymbol{x}+\mathbf{2}, \boldsymbol{x}+\mathbf{4}$ are consecutive even integers.

Consecutive Odd Integers: 7, 9, $\mathbf{1 1}$ are consecutive odd integers. $\boldsymbol{x}, \boldsymbol{x}+\mathbf{2}, \boldsymbol{x}+\mathbf{4}$ are consecutive odd integers.

### 11.3 Examples

Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write
this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line.

Finish by converting all the steps in the problem into symbols. Now look at your preamble and understand it. It will be easier to obtain an equation using the symbols from the preamble.

Write the equation. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

Example 1:
The sum of three consecutive positive integers is 705. Find the integers.

## Solution:

(Method 1)
Preamble:
Let $\boldsymbol{x}$ be the smallest of the three consecutive integers (think of some number, like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then $\boldsymbol{x}+\mathbf{1}$ is the middle number (like $\mathbf{2 0}+\mathbf{1}=\mathbf{2 1}$ ).
And $\boldsymbol{x}+\mathbf{2}$ is the largest number (like $20+2=22$ ).
The sum of the three integers is $\boldsymbol{x}+(\boldsymbol{x}+1)+(\boldsymbol{x}+\mathbf{2})=\mathbf{3 x}+\mathbf{3}$.
Equation:
Sum of integers $=$ sum of integers

$$
\begin{aligned}
3 x+3 & =705 \\
3 x+3-3 & =705-3 \\
3 x & =702 \\
\frac{3 x}{3} & =\frac{702}{3} \\
x & =234
\end{aligned}
$$

Smallest integer: $\boldsymbol{x}=\mathbf{2 3 4}$
Middle integer: $\boldsymbol{x}+\mathbf{1}=\mathbf{2 3 4}+\mathbf{1}=\mathbf{2 3 5}$
Largest integer: $\boldsymbol{x}+\mathbf{2}=\mathbf{2 3 4}+2=236$
(Method 2)
Preamble:
Let $\boldsymbol{x}$ be the middle of the three consecutive integers (think of some number, like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then $\boldsymbol{x}-\mathbf{1}$ is the smallest number (like $20-1=19$ ).
And $\boldsymbol{x}+\mathbf{1}$ is the largest number (like $\mathbf{2 0}+\mathbf{1}=\mathbf{2 1}$ ).
The sum of the three integers is $(\boldsymbol{x}-\mathbf{1})+\boldsymbol{x}+(\boldsymbol{x}+\mathbf{1})=\mathbf{3 x}$.
Equation:

$$
\begin{aligned}
\text { Sum of integers } & =\text { sum of integers } \\
3 \boldsymbol{x} & =\mathbf{7 0 5} \\
\frac{3 x}{3} & =\frac{705}{3} \\
\boldsymbol{x} & =\mathbf{2 3 5}
\end{aligned}
$$

Smallest integer: $\boldsymbol{x} \mathbf{- 1}=\mathbf{2 3 5}-1=\mathbf{2 3 4}$
Middle integer: $\boldsymbol{x}=\mathbf{2 3 5}$
Largest integer: $\boldsymbol{x}+\mathbf{1}=\mathbf{2 3 5}+\mathbf{1}=\mathbf{2 3 6}$
(Method 3)
Preamble:
Let $\boldsymbol{x}$ be the largest of the three consecutive integers (think of some number, like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then $\boldsymbol{x}-\mathbf{1}$ is the middle number (like $20-1=19$ ).
And $\boldsymbol{x}-\mathbf{2}$ is the smallest number (like $20-2=18$ ).
The sum of the three integers is $(\boldsymbol{x}-\mathbf{2})+(\boldsymbol{x}-\mathbf{1})+\boldsymbol{x}=\mathbf{3 x}-\mathbf{3}$.
Equation:
Sum of integers $=$ sum of integers

$$
\begin{aligned}
3 x-3 & =705 \\
3 x-3+3 & =705+3 \\
3 x & =708 \\
\frac{3 x}{3} & =\frac{708}{3} \\
x & =236
\end{aligned}
$$

Smallest integer: $\boldsymbol{x}-2=236-2=234$
Middle integer: $\boldsymbol{x}-\mathbf{1}=\mathbf{2 3 6} \mathbf{- 1}=\mathbf{2 3 5}$
Largest integer: $\boldsymbol{x}=\mathbf{2 3 6}$
Example 2:
Find two integers whose sum is $\mathbf{8 2}$.
11 more than three times the smaller number is the same as 18 less than twice the larger number. Find the numbers.

## Solution:

(Method 1)
Preamble:
Let $\boldsymbol{x}$ be the smaller number (like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the larger number is $\mathbf{8 2 - \boldsymbol { x }}$ (like $\mathbf{8 2 - 2 0}=\mathbf{6 2}$ ).
11 more than three times the smaller number:
$3 x+11$ (like $\mathbf{3}(20)+11$ ).
18 less than twice the larger number:
$2(82-x)-18=164-2 x-18=146-2 x$
(like 2(82-20)-18).
Equation:

11 more than $\mathbf{3}$ times the smaller number $=18$ less than twice the larger number

$$
\begin{aligned}
3 x+11 & =146-2 x \\
3 x+2 x+11 & =146-2 x+2 x \\
5 x+11 & =146 \\
5 x+11-11 & =146-11 \\
5 x & =135 \\
\frac{5}{5} x & =\frac{135}{5} \\
x & =27
\end{aligned}
$$

The smaller number is $\boldsymbol{x}=\mathbf{2 7}$.
The larger number is $\mathbf{8 2 - 2 7}=\mathbf{5 5}$.
(Method 2)
Preamble:
Let $\boldsymbol{x}$ be the larger number (like $\boldsymbol{x}=\mathbf{5 0}$ ).
Then the smaller number is $\mathbf{8 2}-\boldsymbol{x}$ (like $\mathbf{8 2 - 5 0}=\mathbf{6 2}$ ).
11 more than three times the smaller number:
$3(82-x)+11=3(82)-3 x+11=246+11-3 x=257-3 x$ (like $3(82-50)+11)$.

18 less than twice the larger number: $2 \boldsymbol{x}-\mathbf{1 8}$ (like $2(20)-18$ ).
Equation:
11 more than 3 times the smaller number $=18$ less than twice

$$
\begin{aligned}
& \text { the larger number } \\
257-3 x & =2 x-18 \\
257-3 x+3 x & =2 x+3 x-18 \\
257 & =5 x-18 \\
257+18 & =5 x-18+18 \\
275 & =5 x \\
5 x & =275 \\
\frac{5}{5} x & =\frac{275}{5} \\
x & =55
\end{aligned}
$$

The larger number is $\mathbf{5 5}$. The smaller number is $\mathbf{8 2 - 5 5}=\mathbf{2 7}$.
Example 3:
Find two consecutive odd integers such that 60 less than three times the larger number equals $\mathbf{7 3}$ more than twice the smaller number.

## Solution:

(Method 1)

Preamble:
Let $\boldsymbol{x}$ be the smaller odd number (like $\boldsymbol{x}=\mathbf{2 1}$ ).
Then the larger odd number is $\boldsymbol{x}+\mathbf{2}$ (like $\mathbf{2 1}+\mathbf{2}=\mathbf{2 3}$ ).
60 less than 3 times the larger number:
$3(x+2)-60=3 x+6-60=3 x-54$
(like $\mathbf{3}(\mathbf{2 1}+\mathbf{2})-60$ ).
73 more than twice the smaller number: $2 \boldsymbol{2 x} \mathbf{7 3}$ (like $2(21)+73$ ).
Equation:
60 less than $\mathbf{3}$ times the larger number $=73$ more than twice the smaller number

$$
\begin{aligned}
3 x-54 & =2 x+73 \\
3 x-54+54 & =2 x+73+54 \\
3 x & =2 x+127 \\
3 x-2 x & =2 x-2 x+127 \\
x & =127
\end{aligned}
$$

The smaller number is $\boldsymbol{x}=\mathbf{1 2 7}$.
The larger number is $\mathbf{1 2 7}+\mathbf{2}=\mathbf{1 2 9}$.
(Method 2)
Preamble:
Let $\boldsymbol{x}$ be the larger odd number (like $\boldsymbol{x}=\mathbf{2 1}$ ).
Then the smaller odd number is $\boldsymbol{x}-\mathbf{2}$ (like $\mathbf{2 1} \mathbf{- 2}=\mathbf{1 9}$ ).
60 less than three times the larger number:
$3 x-60$ (like $3(21)-60)$.
73 more than twice the smaller number:
$2(x-2)+73=2 x-4+73=2 x+69$ (like $2(21-2)+73)$.
Equation:
60 less than $\mathbf{3}$ times the larger number $=\mathbf{7 3}$ more than twice
the smaller number

$$
\begin{aligned}
3 x-60 & =2 x+69 \\
3 x-60+60 & =2 x+69+60 \\
3 x & =2 x+129 \\
3 x-2 x & =2 x-2 x+129 \\
x & =129
\end{aligned}
$$

The larger odd number is $\mathbf{1 2 9}$. The smaller odd number is $\mathbf{1 2 9 - 2} \mathbf{- 2}$

### 11.4 Exercises 11

1. The sum of three consecutive positive integers is $\mathbf{1 , 1 2 8}$. Find the integers.
2. Find two consecutive even integers such that $\mathbf{1 5 0}$ less than three times the smaller number equals 148 more than twice the larger number.
3. Find two integers whose sum is 425.
$\mathbf{1 0}$ more than six times the smaller number is the same as twice the larger number.
Find the numbers.

## STOP!

1. The sum of three consecutive positive integers is $\mathbf{1 , 1 2 8}$. Find the integers.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the smallest of the three consecutive integers (think of some number, like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then $\boldsymbol{x}+\mathbf{1}$ is the middle number (like $\mathbf{2 0}+\mathbf{1}=\mathbf{2 1}$ ).
And $\boldsymbol{x}+\mathbf{2}$ is the largest number (like $\mathbf{2 0 + 2 = 2 2 )}$.
The sum of the three integers is $x+(x+1)+(x+2)=3 x+3$.
Equation:

$$
\begin{aligned}
\text { Sum of integers } & =\text { sum of integers } \\
3 x+3 & =1,128 \\
3 x+3-3 & =1,128-3 \\
3 x & =1,125 \\
\frac{3 x}{3} & =\frac{1,125}{3} \\
x & =375
\end{aligned}
$$

Smallest integer: $\boldsymbol{x}=\mathbf{3 7 5}$
Middle integer: $\boldsymbol{x + 1}=\mathbf{3 7 5}+\mathbf{1}=\mathbf{3 7 6}$
Largest integer: $\boldsymbol{x}+\mathbf{2}=\mathbf{3 7 5}+\mathbf{2}=\mathbf{3 7 7}$
2. Find two consecutive even integers such that $\mathbf{1 5 0}$ less than three times the smaller number equals 148 more than twice the larger number.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the smaller even integer (like $\boldsymbol{x}=\mathbf{2 1}$ ).
Then the larger even integer is $\boldsymbol{x}+\mathbf{2}$ (like $\mathbf{2 1}+\mathbf{2}=\mathbf{2 3}$ ).
150 less than three times the smaller number: $\mathbf{3 x} \mathbf{- 1 5 0}$ (like $\mathbf{3 ( 2 1 )} \mathbf{- 1 5 0}$ ).
148 more than twice the larger number
$2(x+2)+148=2 x+4+148=2 x+152$
(like $2(21+2)+148)$.
Equation:

150 less than $\mathbf{3}$ times the smaller number $=148$ more than twice
the larger number

$$
\begin{aligned}
3 x-150 & =2 x+152 \\
3 x-2 x-150 & =2 x-2 x+152 \\
x-150 & =152 \\
x-150+150 & =152+150 \\
x & =302
\end{aligned}
$$

The smaller number is $\boldsymbol{x}=\mathbf{3 0 2}$.
The larger number is $\mathbf{3 0 2}+\mathbf{2}=\mathbf{3 0 4}$.
3. Find two integers whose sum is 425.

10 more than six times the smaller number is the same twice as the larger number.
Find the numbers.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the smaller number
(like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the larger number is $425-x$ (like $425-20=405$ ).
$\mathbf{1 0}$ more than six times the smaller number: $\mathbf{6 x}+\mathbf{1 0}$
(like $\mathbf{6 ( 2 0 )}+\mathbf{1 0}$ ).
twice the larger number: $2(425-x)=850-2 x$
(like 2(425-20)).
Equation:
10 more than 6 times the smaller number $=$ twice the larger number
$6 x+10=850-2 x$
$6 x+2 x+10=850-2 x+2 x$
$8 x+10=850$
$8 x+10-10=850-10$
$8 x=840$
$\frac{8}{8} x=\frac{840}{8}$
$x=105$
The smaller number is $\boldsymbol{x}=\mathbf{1 0 5}$.
The larger number is $425-105=320$.

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## Chapter 12

## Solving Linear Equations. Coin and Stamps Problems

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### 12.1 Youtube

http://www.youtube.com/playlist?list=PL8DEEB788515A427F\&feature=view_all

### 12.2 Basics

A coin (or a stamp) has a value. A dime is worth 10 cents.
The total value of a number of coins (or stamps) is the product of the number and the value.

### 12.3 Examples

Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble.

Now start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

Example 1:
A coin bank contains nickels, dimes and quarters.
There are $\mathbf{8}$ more dimes than quarters.
The number of nickels is $\mathbf{8}$ less than twice the number of dimes.
The total value of the coins is $\$ 5.70$.
Find the number of each coin.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of quarters (think of some number, like $\boldsymbol{x}=\mathbf{2 0}$ quarters).
There are $\mathbf{8}$ more dimes than quarters: $\boldsymbol{x}+\mathbf{8}$
(like $\mathbf{2 0}+\mathbf{8}=\mathbf{2 8}$ dimes).
The number of nickels is $\mathbf{8}$ less than twice the number of dimes: $2(x+8)-\mathbf{8}=\mathbf{2 x}+\mathbf{1 6 - 8}=\mathbf{2 x}+\mathbf{8}$ (like $2(20+8)-8)$.

| Coin | Number of <br> coins in cents | Value of <br> one coin | Total value <br> of the coins |
| :--- | :---: | :---: | :---: |
| Nickels | $\mathbf{2 x}+\mathbf{8}$ | $\mathbf{5}$ | $\mathbf{5 ( 2 \boldsymbol { x } + \mathbf { 8 } )}$ |
| Dimes | $\boldsymbol{x}+\mathbf{8}$ | $\mathbf{1 0}$ | $\mathbf{1 0}(\boldsymbol{x}+\mathbf{8})$ |
| Quarters | $\boldsymbol{x}$ | $\mathbf{2 5}$ | $\mathbf{2 5 x}$ |

Equation:

$$
\text { The total value of the coins }=\$ 5.70=\mathbf{5 7 0} \text { cents }
$$

$$
\begin{aligned}
5(2 x+8)+10(x+8)+25 x & =570 \\
10 x+40+10 x+80+25 x & =570 \\
45 x+40+80 & =570 \\
45 x+120 & =570 \\
45 x+120-120 & =570-120 \\
45 x & =450 \\
\frac{45 x}{45} & =\frac{450}{45} \\
x & =10
\end{aligned}
$$

Number of quarters: $\boldsymbol{x}=\mathbf{1 0}$
Number of dimes: $x+8=10+8=18$
Number of nickels: $2 \boldsymbol{x}+\mathbf{8}=\mathbf{2 0}+\mathbf{8}=\mathbf{2 8}$
Note: If $\boldsymbol{x}$ had been chosen to be the number of dimes,
the number of quarters is $\mathbf{8}$ less than the number of dimes. We have to rewrite the sentence "there are $\mathbf{8}$ more dimes than quarters" as "there are $\mathbf{8}$ less quarters than dimes" which is not a good idea.

If $\boldsymbol{x}$ had been chosen to be the number of nickels, the number of dimes is half more than the sum of $\mathbf{8}$ and the number of nickels. Wow! This is not easy to figure out. Letting $\boldsymbol{x}$ be the number of quarters was the best choice.

## Example 2:

A wallet contains 42 one-, five- and ten-dollar bills.
The value of all the bills is $\mathbf{\$ 1 3 5}$.
There are 5 times as many one-dollar bill as ten-dollar bills.
Find the number of bills of each denomination.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of $\mathbf{\$ 1 0}$ bills (like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the number of $\$ \mathbf{1}$ bills is $\mathbf{5 x}$ (like $\mathbf{5 ( 2 0 )}$.
A wallet contains 42 bills (including ??? five-dollar bills):

$$
\begin{aligned}
x+5 x+? ? ? & =42 \\
6 x+? ? ? & =42 \\
6 x-6 x+? ? ? & =42-6 x \\
? ? ? & =42-6 x
\end{aligned}
$$

The number of $\$ 5$ is $? ? ?=42-6 x$

| Coin | Number of <br> coins in cents | Value of <br> one coin | Total value <br> of the coins |
| :--- | :---: | :---: | :---: |
| $\$ 10$ | $\boldsymbol{x}$ | $\mathbf{1 0}$ | $\mathbf{1 0 x}$ |
| $\$ \mathbf{5}$ | $\mathbf{4 2 - \mathbf { 6 x }}$ | $\mathbf{5}$ | $\mathbf{5 ( 4 2 - \mathbf { 6 x } )}$ |
| $\$ \mathbf{1}$ | $\mathbf{5 x}$ | $\mathbf{1}$ | $\mathbf{5 x}$ |

Equation:

$$
\begin{aligned}
\text { The total value of the coins } & =\$ 135 \text { dollars } \\
10 x+5(42-6 x)+5 x & =135 \\
10 x+210-30 x+5 x & =135 \\
210-15 x & =135 \\
210-210-15 x & =135-210 \\
-15 x & =\frac{-75}{-15 x} \\
\frac{-75}{-15} & =\frac{-15}{x}
\end{aligned}
$$

The number of ten-dollar bills is $\mathbf{5}$.
The number of five-dollar bills is
$42-6 x=42-6(5)=42-30=12$.
The number of one-dollar bills is $\mathbf{5 ( 5 )}=\mathbf{2 5}$.
Example 3:

An order of $\mathbf{\$ 0 . 4 4}$ stamps and $\$ \mathbf{1 . 2 9}$ custom photo postcards cost $\$ \mathbf{1 4 . 8 7}$. The order consists of $\mathbf{2 8}$ items. Find the number of custom photo postcards.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of stamps (like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the number postcards is $\mathbf{2 8}-\boldsymbol{x}$ (like $\mathbf{2 8} \mathbf{- 2 0}$ ).

| Stamps <br> or postcards | Number of <br> items | Value of <br> one item | Total value <br> of the items |
| :--- | :---: | :---: | :---: |
| stamp | $\boldsymbol{x}$ | $\mathbf{0 . 4 4}$ | $\mathbf{0 . 4 4 \boldsymbol { x }}$ |
| postcard | $\mathbf{2 8}-\boldsymbol{x}$ | $\mathbf{1 . 2 9}$ | $\mathbf{1 . 2 9 ( 2 8 - \boldsymbol { x } )}$ |

## Equation:

$$
\begin{aligned}
\text { The whole order costs } & =\$ 14.87 \\
0.44 x+1.29(28-x) & =14.87 \\
0.44 x+36.12-1.29 x & =14.87 \\
-0.85 x+36.12 & =14.87 \\
-0.85 x+36.12-36.12 & =14.87-36.12 \\
-0.85 x & =-21.25 \\
\frac{-0.85}{-0.85} x & =\frac{-21.25}{-0.85} \\
x & =25
\end{aligned}
$$

The order consists of $\mathbf{2 5}$ stamps and $\mathbf{2 8} \mathbf{- 2 5}=\mathbf{3}$ postcards.

### 12.4 Exercises 12

1. A collection of stamps from a foreign country consists of five-franc stamps and two-franc stamps. The number of two-franc stamps is 12 less than twice two-franc stamps.
The total value of the stamps is $\$ \mathbf{3 8 1}$.
Find the number of stamps.
2. A cash register contains five-franc, ten-franc and twenty-five-franc coins.

The number of five-franc coins is twice the number of ten-franc coins.
The number of twenty-five-franc coins is 4 more than the number of five-franc coins. The total value of the coins is $\mathbf{9 4 0}$ francs. How many coins of each denomination are there?

## STOP!

1. A collection of stamps from a foreign country consists of five-franc stamps and two-franc stamps.

The number of two-franc stamps is 12 less than twice five-franc stamps.
The total value of the stamps is $\mathbf{3 8 1}$ francs. Find the number of stamps.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of five-franc stamps (like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the number two-franc stamps is $\mathbf{1 2}$ less than twice five-franc stamps: $\mathbf{2 x} \mathbf{x} \mathbf{1 2}$ (like $2(20)-12)$.

| Stamp | Number of <br> stamps | Value of <br> one stamp | Total value <br> of the stamps |
| :--- | :---: | :---: | :---: |
| five-franc | $\boldsymbol{x}$ | $\mathbf{5}$ | $\mathbf{5 x}$ |
| two-franc | $\mathbf{2 x}-\mathbf{1 2}$ | $\mathbf{2}$ | $\mathbf{2 ( 2 \boldsymbol { x }}-\mathbf{1 2 )}$ |

Equation:
The total value of the stamps $=\mathbf{3 8 1}$ francs

$$
\begin{aligned}
5 x+2(2 x-12) & =381 \\
5 x+4 x-24 & =381 \\
9 x-24 & =381 \\
9 x-24+24 & =381+24 \\
9 x & =405 \\
\frac{9}{9} x & =\frac{405}{9} \\
x & =45
\end{aligned}
$$

The number of five-franc stamps is 45 .
The number of two-franc stamps is
$2(45)-12=90-12=78$.
2. A cash register contains five-franc, ten-franc and twenty-five-franc coins.

The number of five-franc coins is twice the number of ten-franc coins.
The number of twenty-five-franc coins is $\mathbf{4}$ more than the number of five-franc coins.
The total value of the coins is $\mathbf{9 4 0}$ francs. How many coins of each denomination are there?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of ten-franc coins (like $\boldsymbol{x}=\mathbf{2 0}$ ).
Then the number of five-franc coins is twice the number of ten-franc coins: $\mathbf{2 x}$ (like $\mathbf{2 ( 2 0 )}=\mathbf{4 0}$ ).
The number of twenty-five-franc coins is $\mathbf{4}$ more than the number of five-franc coins: $\mathbf{2 x}+\mathbf{4}$ (like $2(20)+4)$

| Coin | Number of <br> coins | Value of <br> one coin | Total value <br> of the coins |
| :--- | :---: | :---: | :---: |
| five-franc | $\mathbf{2 x}$ | $\mathbf{5}$ | $\mathbf{5 ( 2 \boldsymbol { x } )}$ |
| ten-franc | $\boldsymbol{x}$ | $\mathbf{1 0}$ | $\mathbf{1 0} \boldsymbol{x}$ |
| twenty-five | $\mathbf{2 x}+\mathbf{4}$ | $\mathbf{2 5}$ | $\mathbf{2 5 ( 2 \boldsymbol { x } + \mathbf { 4 ) }}$ |

Equation:

$$
\begin{aligned}
\text { The total value of the coins } & =\$ 940 \text { francs } \\
5(\mathbf{2 x})+\mathbf{1 0 x}+\mathbf{2 5}(\mathbf{2 x}+\mathbf{4 )} & =\mathbf{9 4 0} \\
\mathbf{1 0 x + 1 0 x + 5 0 x + 1 0 0} & =\mathbf{9 4 0} \\
\mathbf{7 0 x}+\mathbf{1 0 0} & =\mathbf{9 4 0} \\
\mathbf{7 0 x}+\mathbf{1 0 0}-\mathbf{1 0 0} & =\mathbf{9 4 0}-\mathbf{1 0 0} \\
\mathbf{7 0 x} & =840 \\
\frac{\mathbf{7 0 x}}{\mathbf{7 0}} & =\frac{\mathbf{1 0 4 0}}{\mathbf{7 0}} \\
\boldsymbol{x} & =\mathbf{1 2}
\end{aligned}
$$

The number of five-franc coins is $\mathbf{2 ( 1 2 )}=\mathbf{2 4}$.
The number of ten-franc coins is $\mathbf{1 2}$.
The number of twenty-five-franc coins is $\mathbf{2 4}+\mathbf{4}=\mathbf{2 8}$.

## Chapter 13

## Solving Linear Equations. Geometry Problems

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### 13.1 Youtube

http://www.youtube.com/playlist?list=PL449A4F29D1F85987\&feature=view_all

### 13.2 Recommendation

Where possible draw a sketch of the figure(s) involved in the problem.
Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

### 13.3 Examples

## Example 1:

The perimeter of a rectangle is $\mathbf{6 8} \mathrm{ft}$. Its length is $\mathbf{7}$ more than twice its width. Find the dimensions of the rectangle.

## Solution:

Preamble:
$\boldsymbol{x}$ is the width $\boldsymbol{W}$.

$\boldsymbol{2 x}+\mathbf{7}$ is the length $\boldsymbol{L}$.
Equation:
The perimeter is

$$
\begin{array}{rlr}
P & =2(L+W) & \\
68 & =2(2 x+7+x) & \\
68 & =2(3 x+7) & \\
68 & =6 x+14 & \\
68-14 & =6 x+14-14 & \\
54 & =6 x \\
\frac{54}{6} & =\frac{6}{6} x & 9 \\
9 & =x & 2(9)+7=18+7=25
\end{array}
$$

The dimensions of the rectangle are $\mathbf{2 5} \mathrm{ft}$ and $\mathbf{9} \mathrm{ft}$.
Example 2:
Angle $\mathbf{1}$ of a triangle is $\mathbf{1 1}$ more than three times angle 2. Angle $\mathbf{3}$ is $\mathbf{1 5}$ less than $\mathbf{4}$ times angle 2. Find the measure of all three angles.
Then find the complement and the supplement of the largest angle.
Hint: The sum of the angles of a triangle is $\mathbf{1 8 0}^{\circ}$.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the measure of angle $\mathbf{2}$.
Angle $\mathbf{1}$ of is $\mathbf{1 1}$ more than three times angle 2: $\mathbf{3 x}+\mathbf{1 1}$.
Angle $\mathbf{3}$ is 15 less than 4 times angle 2: $\mathbf{4 x} \mathbf{- 1 5}$.


Equation:

$$
\begin{aligned}
\text { The sum of the angles } & =180^{\circ} \\
x+(3 x+11)+(4 x-15) & =180 \\
x+3 x+11+4 x-15 & =180 \\
8 x-4 & =180 \\
8 x-4+4 & =180+4 \\
8 x & =184 \\
\frac{8}{8} x & =\frac{184}{8} \\
x & =23
\end{aligned}
$$

Angle $2=23^{\circ}$
Angle $1=3(23)+11=69+11=80^{\circ}$
Angle $3=4(23)-15=92-15=77^{\circ}$
The complement of $\mathbf{8 0}{ }^{\circ}$ is $\mathbf{9 0}-\mathbf{8 0}=\mathbf{1 0}^{\circ}$.
The supplement of $\mathbf{8 0}{ }^{\circ}$ is $\mathbf{1 8 0}-\mathbf{8 0}=\mathbf{1 0 0}^{\circ}$.


Example 3:
Two lines intersect. One of the acute angles is $\mathbf{4 5}{ }^{\circ}$. Find the measures of the other three angles.

## Solution:

Preamble:
Equation:


The four angles make a complete revolution $\left(\mathbf{3 6 0}^{\circ}\right)$. $\boldsymbol{A}$ added to $\boldsymbol{B}$ measures a half revolution or $\mathbf{1 8 0}^{\circ}$.
Similarly $B+45=180^{\circ}$. Thus $B=180-45=135^{\circ} . B$ is the supplement of $45^{\circ}$.
$C+45=180^{\circ}$. Thus $C=180-45=135^{\circ} . C$ is the supplement of $45^{\circ}$.
$C+A=180^{\circ}$. Thus $A=180-C=180-135=45^{\circ}$.
The pair of angles $45^{\circ}$ and $\boldsymbol{C}$ are said to be vertical angles.
The pair of angles $\boldsymbol{A}$ and $\boldsymbol{B}$ are also called vertical angles.
Example 4:
Two parallel lines are intersected by a transversal.

Find angles $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$.


## Solution:

$\mathbf{4 0 ^ { \circ }}$ added to $\boldsymbol{A}$ is $\mathbf{1 8 0}^{\circ}$. $\boldsymbol{A}$ is the supplement of $\mathbf{4 0 ^ { \circ }}$ or $\mathbf{1 4 0 ^ { \circ }}$.
The $\mathbf{4 0 ^ { \circ }}$ angle and $\boldsymbol{B}$ are alternate interior angles. Their measures are equal. Thus $\boldsymbol{B}=\mathbf{4 0 ^ { \circ }}$.
$\boldsymbol{B}$ and $\boldsymbol{C}$ are vertical angles. These angles are congruent (equal in all respects). Thus $\boldsymbol{C}=\mathbf{4 0 ^ { \circ }}$.


The space between the two vertical lines is the interior of the parallel lines. The region above the top line and the region below the bottom line is the exterior.
$180^{\circ}$ is said to be a straight angle.
$\mathbf{9 0}{ }^{\circ}$ is called a right angle.
$\mathbf{4 0 ^ { \circ }}$ is an acute angle because it is $<\mathbf{9 0}^{\circ}$.
$135^{\circ}$ is an obtuse angle because it is between $90^{\circ}$ and $180^{\circ}$.

Example 5:
Find the measures of the angles in the given figure.

## Solution:

The two adjacent angles form a straight angle. The sum of their measures is $\mathbf{1 8 0}^{\circ}$.

$$
\begin{aligned}
&(2 x+47)+(x-17)=180 \\
& 2 x+47+x-17=180 \\
& 3 x+30=180 \\
& 3 x+30-30=180-30 \quad \underline{2 x+47} x-17 \\
& 3 x=150 \\
& 3 x=\frac{150}{3} \\
& x=50 \\
& 2(50)+47=100+47=147^{\circ}
\end{aligned}
$$

Example 6:
An isosceles triangle has two congruent sides. One of these is $\mathbf{2}$ more than double the non-congruent side. Find the dimensions of the triangle if the perimeter is $\mathbf{1 7 4} \mathrm{ft}$.

## Solution:

Preamble:
Equation:


$$
\begin{aligned}
\text { Perimeter } & =\text { Perimeter } \\
x+(2 x+2)+(2 x+2) & =174 \\
x+2 x+2+2 x+2 & =174 \\
5 x+4 & =174 \\
5 x+4-4 & =174-4 \\
5 x & =170 \\
\frac{5}{5} x & =\frac{170}{5} \Rightarrow x=34
\end{aligned}
$$

The dimensions of the triangle are $\mathbf{7 0}, \mathbf{7 0}$, and $\mathbf{3 4} \mathrm{ft}$.
Example 7:
Triangle $\boldsymbol{A B C}$ is shown on the right. Trapezoid $\boldsymbol{A D E C}$ is inscribed in the triangle. Which has the greater perimeter? The triangle or the trapezoid?


The question boils down to triangle $\boldsymbol{B D} \boldsymbol{E}$.

Equation:


Line segments $\boldsymbol{D} \boldsymbol{A}, \boldsymbol{A C}$ and $\boldsymbol{C E}$ are part of both the triangle and the trapezoid.
Side $\boldsymbol{D} \boldsymbol{E}$ which completes the trapezoid is shorter than the sum of $\boldsymbol{D} \boldsymbol{B}$ and $\boldsymbol{E} \boldsymbol{B}$ which complete the triangle.

The triangle has the greater perimeter.
Example 8:
Side $\mathbf{1}$ of a triangle is $\mathbf{2} \mathrm{ft}$ more than side $\mathbf{2}$. Side $\mathbf{3}$ is $\mathbf{2} \mathrm{ft}$. less than side $\mathbf{2}$.
The side of a square is $\mathbf{9} \mathrm{ft}$. The perimeter of the triangle and the square is the same number. What are the dimensions of the triangle?


Let $\boldsymbol{x}$ be side $\mathbf{2}$. Then $\boldsymbol{x}+\mathbf{2}$ and $\boldsymbol{x}-\mathbf{2}$ are the other sides of the triangle
Perimeter of triangle:
$(x-2)+x+(x+2)=x-2+x+x+2=3 x$
Perimeter of square: $\mathbf{4 ( 9 )}=36$

Equation:
Perimeter of triangle $=$ Perimeter of square

$$
\begin{aligned}
3 x & =36 \\
\frac{3}{3} x & =\frac{36}{3} \\
x & =12
\end{aligned}
$$

The dimensions of the triangle are

$12-2=10,12$ and $12+2=14 \mathrm{ft}$.

Example 9:
Find the three angles in the figure on the right.


Preamble:
The sum of the three angles is a straight angle.


Equation:

$$
\begin{aligned}
\text { The sum of the three angles } & =180^{\circ} \\
(5 x+5)+5 x+(3 x+6) & =180 \\
5 x+5+5 x+3 x+6 & =180 \\
13 x+5+6 & =180 \\
13 x+11 & =180 \\
13 x+11-11 & =180-11 \\
13 x & =169 \\
\frac{13}{13} x & =\frac{169}{13} \\
x & =13
\end{aligned}
$$

$5 x+5=5(13)+5=65+5=70$,
$5 x=5(13)=65$,
$3 x+6=3(13)+6=39+6=45$.

### 13.4 Exercises 13

1. The perimeter of a rectangular garden is $\mathbf{1 0 4} \mathrm{ft}$. Its length is $\mathbf{8} \mathrm{ft}$ less than twice its width. Find the dimensions of the rectangle.
2. Angle $\mathbf{1}$ of a triangle is $\mathbf{2}$ more than four times angle $\mathbf{2}$. Angle $\mathbf{3}$ is $\mathbf{9}$ less than $\mathbf{6}$ times angle $\mathbf{2}$. Find the measure of all three angles.
Then find the complement and the supplement of the middle angle.
Hint: The sum of the angles of a triangle is $\mathbf{1 8 0}^{\circ}$.
3. Two rural roads intersect. One of the acute angles formed by the roads is $\mathbf{5 3}{ }^{\circ}$. Find the measures of the other three angles.

4. A pair of railroad tracks (parallel lines) is intersected by a road (transversal). A gate at $\boldsymbol{C}$ is to swing between the track and the road. Through how many degrees should the gate swing? (What is the measure of $\boldsymbol{C}$ ?)

5. A bridge crosses a river at an angle. Angular building 1 sits on the river's edge on one side of the bridge. Angular building 2 is constructed on the river's edge on the other side of the bridge. The angle for building $\mathbf{2}$ is twice that of building $\mathbf{1}$. The angles left between the buildings and the bridge are shown in the figure below. Find the measure of the acute angle between the bridge and the river.

6. A $\log$ is in a horizontal position. It is suspended by two cables joined at a point to form an isosceles triangle with the log as one side of the triangle. Each cable is $\mathbf{5} \mathrm{ft}$ less than $\mathbf{3}$ times the length of the $\log$. Find the length of one cable if the perimeter is $\mathbf{9 5} \mathrm{ft}$.
7. A block of wood has a cross section in the form of an equilateral (all three sides are congruent) triangle $\boldsymbol{A B C}$ as shown on the right. Each side is $\mathbf{1 0} \mathrm{ft}$ long. Trapezoid $\boldsymbol{A} \boldsymbol{D E C}$ is formed by cutting off the top half of the triangle. What is the length of the top $\boldsymbol{D} \boldsymbol{E}$ of the trapezoid?

8. A pile of logs is stacked in the triangular shape shown in the triangle on the right.

Side $\mathbf{1}$ of a triangle is $\mathbf{3} \mathrm{ft}$ more than side $\mathbf{2}$. Side $\mathbf{3}$ is $\mathbf{3} \mathrm{ft}$. less than side $\mathbf{2}$.
The side of a square is $\mathbf{1 5} \mathrm{ft}$. The perimeter of the triangle and the square is the same number. What are the dimensions of the triangle?

9. Two searchlights are programmed to illuminate the sky at angles configured in the picture on the right. Find the three angles.


## STOP!

1. The perimeter of a rectangular garden is $\mathbf{1 0 4} \mathrm{ft}$. Its length is $\mathbf{8} \mathrm{ft}$ less than twice its width. Find the dimensions of the rectangle.

## Solution:

Preamble:
$\boldsymbol{x}$ is the width $\boldsymbol{W}$.
$2 \boldsymbol{x}-\mathbf{8}$ is the length $L$.
Equation:


The perimeter is

$$
\begin{aligned}
P & =2(L+W) \\
104 & =2(2 x-8+x) \\
104 & =2(3 x-8) \\
104 & =6 x-16 \\
104+16 & =6 x-16+16 \\
120 & =6 x \\
\frac{120}{6} & =\frac{6}{6} x \\
20 & =x \\
x & =20
\end{aligned}
$$

The dimensions of the rectangle are $\mathbf{3 2}$ by $\mathbf{2 0} \mathrm{ft}$.

| $2(20)-8=40-8=32$ |
| ---: |

2. Angle $\mathbf{1}$ of a triangle is $\mathbf{2}$ more than four times angle 2. Angle $\mathbf{3}$ is $\mathbf{9}$ less than $\mathbf{6}$ times angle 2. Find the measure of all three angles.
Then find the complement and the supplement of the middle angle.
Hint: The sum of the angles of a triangle is $\mathbf{1 8 0}^{\circ}$.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the measure of angle $\mathbf{2}$.
Angle 1 of is $\mathbf{2}$ more than four times angle 2: $\mathbf{4 x}+\mathbf{2}$.
Angle $\mathbf{3}$ is $\mathbf{9}$ less than $\mathbf{6}$ times angle 2: $\mathbf{6 x - 9}$.
Equation:
The sum of the angles $=180^{\circ}$
$x+(4 x+2)+(6 x-9)=180$
$x+4 x+2+6 x-9=180$
$11 x-7=180$
$11 x-7+7=180+7$
$11 x=187$
$\frac{11}{11} x=\frac{187}{11}$
$x=17$
Angle $2=17^{\circ}$
Angle $1=4(17)+2=68+2=70^{\circ}$
Angle $3=6(17)-9=102-9=93^{\circ}$


The complement of $\mathbf{7 0}{ }^{\circ}$ is $\mathbf{9 0}-\mathbf{7 0}=\mathbf{2 0}{ }^{\circ}$.
The supplement of $\mathbf{7 0}{ }^{\circ}$ is $\mathbf{1 8 0} \mathbf{- 7 0}=\mathbf{1 1 0}^{\circ}$.
3. Two rural roads intersect. One of the acute angles formed by the roads is $\mathbf{5 3}{ }^{\circ}$. Find the measures of the other three angles.

## Solution:

Preamble:


Equation:
The four angles make a complete revolution $\left(\mathbf{3 6 0}{ }^{\circ}\right)$. $\boldsymbol{A}$ added to $\boldsymbol{B}$ measures a half revolution or $180^{\circ}$.

Similarly $B+53=180^{\circ}$. Thus $B=180-53=127^{\circ}$. $B$ is the supplement of $53^{\circ}$.
$C+53=180^{\circ}$. Thus $C=180-\mathbf{5 3}=127^{\circ} . C$ is the supplement of $53^{\circ}$.
$C+A=180^{\circ}$. Thus $A=180-C=180-127=\overline{53^{\circ}}$.
4. A pair of railroad tracks (parallel lines) is intersected by a road (transversal). A gate at $\boldsymbol{C}$ is to swing between the track and the road. Through


SBTutiont: degrees should the gate swing?
$\mathbf{8 0}{ }^{\circ}$ added to $\boldsymbol{A}=\mathbf{1 8 0 ^ { \circ }} . \boldsymbol{A}=\mathbf{1 0 0}$ is the supplement of $\mathbf{8 0 ^ { \circ }}$.
$\mathbf{8 0}{ }^{\circ}$ and $C$ are alternate interior angles. Their measures are equal. Thus $C=80^{\circ}$.
The gate should swing through $80^{\circ}$.
5. A bridge crosses a river at an angle. Angular building 1 sits on one side of the bridge. Angular building 2 is constructed on the other side of the bridge. The angle for building $\mathbf{2}$ is twice that of building 1. The angles left between the buildings and the bridge are shown in the figure on the right. Find the measure of the acute angle between the bridge and the river.


## Solution:

The two adjacent angles on the river's edge form a straight angle. The sum of their measures is $\mathbf{1 8 0}^{\circ}$.

$$
\begin{aligned}
(2 x+36)+(x+18) & =180 \\
2 x+72+x+18 & =180 \\
3 x+90 & =180 \\
3 x+90-90 & =180-90 \\
3 x & =90 \\
\frac{3}{3} x & =\frac{90}{3} \\
x & =30
\end{aligned}
$$



The angle between the bridge and the road is $48^{\circ}$
6. A $\log$ is in a horizontal position. It is suspended by two cables joined at a point to form an isosceles triangle with the log as one side of the triangle. Each cable is $\mathbf{5} \mathrm{ft}$ less than $\mathbf{3}$ times the length of the log. Find the length of one cable if the perimeter is $\mathbf{9 5} \mathrm{ft}$.

## Solution:

Preamble:
See the picture.
Equation:

$$
\begin{aligned}
\text { Perimeter } & =\text { Perimeter } \\
x+(3 x-5)+(3 x-5) & =95 \\
x+3 x-5+3 x-5 & =95 \\
7 x-10 & =95 \\
7 x-10+10 & =95+10 \\
7 x & =105 \\
\frac{7}{7} x & =\frac{105}{7} \\
x & =15
\end{aligned}
$$



Each cable is 40 ft long.
7. A block of wood has a cross section in the form of an equilateral (all three sides are congruent) triangle $\boldsymbol{A B C}$ as shown on the right. Each side is $\mathbf{1 0} \mathrm{ft}$ long. Trapezoid $\boldsymbol{A D E C}$ is formed by cutting off the top half of the triangle. What is the length of the top $\boldsymbol{D E}$ of the trapezoid?

## Solution:

The question boils down to triangle $\boldsymbol{B} \boldsymbol{D} \boldsymbol{E}$. This triangle is also an equilateral triangle. So side $D E=5 \mathrm{ft}$.
8. A pile of logs is stacked in the triangular shape shown in the triangle on the right.

Side $\mathbf{1}$ of a triangle is $\mathbf{3} \mathrm{ft}$ more than side $\mathbf{2}$. Side $\mathbf{3}$ is $\mathbf{3} \mathrm{ft}$. less than side $\mathbf{2}$.
The side of a square is $\mathbf{1 5} \mathrm{ft}$. The perimeter of the triangle and the square is the same number.
What are the dimensions of the triangle?

## Solution:

Preamble:


Let $\boldsymbol{x}$ be side $\mathbf{2}$. Then $\boldsymbol{x}+\mathbf{3}$ and $\boldsymbol{x}-\mathbf{3}$ are the other sides of the triangle
Perimeter of triangle:
$(x-3)+x+(x+3)=x-3+x+x+3=3 x$
Perimeter of square: $\mathbf{4 ( 1 5 )}=\mathbf{6 0}$
Equation:
Perimeter of triangle $=$ Perimeter of square

$$
3 x=60
$$



$$
\frac{3}{3} x=\frac{60}{3}
$$

$$
x=20
$$

The dimensions of the triangle are

$$
20-3=17, \quad 20 \text { and } 20+3=23 \mathrm{ft}
$$

9. Two searchlights are programmed to illuminate the sky at angles configured in the picture on the right. Find the three angles.

## Solution

Preamble:
searchlight $2 \quad$ searchlight 1


The sum of the three angles is a straight angle.
Equation:

$$
\begin{aligned}
& \text { The sum of the three angles }=18 \mathbf{0}^{\circ} \\
& (5 x-6)+(x+7)+(3 x+17)=180 \\
& 5 x-6+x+7+3 x+17=180 \\
& 9 x-6+7+17=180 \\
& 9 x+18=180 \\
& 9 x+18-18=180-18 \\
& 9 x=162 \\
& \frac{9}{9} x=\frac{162}{9} \\
& x=18 \\
& 5 x-6=5(18)-6=90-6=84 \text {, } \\
& x+7=18+7=25 \text {, } \\
& 3 x+17=3(18)+17=54+17=71 \text {. }
\end{aligned}
$$



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## Chapter 14

## Solving Linear Equations. Commerce Problems

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### 14.1 Youtube

http://www.youtube.com/playlist?list=PL176B9B783E43747B\&feature=view_all
Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

### 14.2 Markup and Discount

A retailer buys 20 pairs of shoes from a manufacturer. The retailer has expenses. He needs to travel, probably pay rent, utilities, taxes, etc. To cover his/her expenses the retailer adds a fraction of the cost of a pair of shoe to the cost when the pair is sold. This is profit used to pay expenses. That fraction, usually a percent of the cost, is called markup.

Now suppose the shoes don't sell. Maybe the fashion changes, the economy worsens, the clientele migrates, etc. The retailer has to get rid of the shoes. As an incentive to entice customers to buy, the retailer takes off a fraction (a percent) of the before-discount price, that is the retail price. This fraction is called discount. The sales price is the price at which the shoes will sell after application of the discount. The discount is sometimes called a markdown.

The phrase"Markup is percent(decimal) of Cost", with emphasis on is and of, will be quite beneficial.

### 14.3 Examples

Example 1:
The Pedstore buys $\mathbf{1 6}$ pairs of shoes at a cost of $\$ \mathbf{2 7}$ each. The markup rate is $\mathbf{4 0 \%}$. What is the markup and what is the sales price?

## Solution:

Preamble:
Let $\boldsymbol{S}$ be the sales price.
The cost is $\$ \mathbf{2 7}$ and the markup rate is $\mathbf{4 0 \%}=\mathbf{0 . 4}$.
Equation:

$$
\begin{aligned}
\text { Markup } & =\text { percent }(\text { decimal }) \text { of Cost } \\
\boldsymbol{M} & =(\mathbf{0 . 4})(\mathbf{2 7}) \\
& =\mathbf{1 0 . 8}
\end{aligned}
$$

The markup is $\mathbf{\$ 1 0 . 8 0}$. The sales price is
$27.00+10.80=\$ 37.80$.
Note: The markup is $\mathbf{0 . 4}$ for every dollar of cost. Then the sales price is $\mathbf{1 . 0}+\mathbf{0 . 4}=\mathbf{1 . 4}$ dollars.
We could have found the sales price more quickly by multiplying 27 by 1.4 .
Check: $\mathbf{1 . 4} \cdot \mathbf{2 7}=\mathbf{3 7 . 8 0}$
Example 2:
The Pedstore buys $\mathbf{1 6}$ pairs of shoes at a cost of $\$ \mathbf{3 5}$ each. Each pair is on sale for $\mathbf{\$ 6 5}$. Find the markup rate (as a percent). Round to the nearest percent.

## Solution:

Preamble:

Let $\boldsymbol{x}$ be the markup rate (as a decimal).
The cost is $\$ \mathbf{3 5}$, the sales price is $\$ \mathbf{6 5}$. The markup is $65-35=\$ 30$

Equation:

$$
\begin{aligned}
\text { Markup } & =\text { percent(decimal) of Cost } \\
\mathbf{3 0} & =(\boldsymbol{x})(\mathbf{3 5}) \\
\frac{\mathbf{3 0}}{\mathbf{3 5}} & =\frac{\mathbf{3 5}}{\mathbf{3 5}} \boldsymbol{x} \\
\frac{\mathbf{6}}{\mathbf{7}} & =\boldsymbol{x} \\
\boldsymbol{x} & =\mathbf{0 . 8 5 7 1} \\
\boldsymbol{x} & =\mathbf{8 5 . 7 1 \%}
\end{aligned}
$$

The markup rate is $\mathbf{8 6 \%}$.
Check: $1.86 \cdot \mathbf{3 5}=\mathbf{6 5 . 1}$ (the extra $\mathbf{1 0}$ cents is the result of rounding $\mathbf{8 5 . 7 1}$ to $\mathbf{8 6}$.)
Example 3:
The Pedstore sells a pair of shoes for $\mathbf{\$ 8 9 . 9 9}$. The markup rate is $\mathbf{2 5 \%}$. What was the cost of a pair of shoes? (Round to the nearest cent.)

## Solution:

Preamble:
Let $\boldsymbol{C}$ be the cost. Then the markup is $\boldsymbol{M}=\mathbf{0 . 2 5 C}$ and the sales price is $\boldsymbol{S}=\boldsymbol{C}+\mathbf{0 . 2 5 C = 1 . 2 5 C}$.
Equation:

$$
\begin{aligned}
\text { Sales price } & =\text { Sales price } \\
1.25 C & =89.99 \\
\frac{1.25}{1.25} C & =\frac{89.99}{1.25} x \\
C & =71.99 \\
C & =\$ 72
\end{aligned}
$$

Check: $M=\mathbf{0 . 2 5 ( 7 2 )}=\$ 18, S=18+72=90$.

Example 4:
A $\mathbf{\$ 1 5 9 . 9 9}$ (Regular price) dress has been sitting in at Elegante Fashion window for over a year. The merchant encourages you to buy the dress after applying a $\mathbf{1 5 \%}$ discount. What are you buying the dress for?

## Solution:

Preamble:
Let $\boldsymbol{S}$ be the sales price. The discount is $\mathbf{1 5 \%}$ of the regular price.
Equation:
Discount $=$ discount rate of regular price

$$
D=(0.15)(159.99)
$$

$$
=24
$$

The sales price $S=159.99-24=\$ 135.99$
Example 5:
The regular price of a hat in Elegante Fashion window is $\mathbf{3 4 . 9 9}$ (regular price). It is discounted to sell for 27.99. What is the discount rate (in percent)? Round to the nearest percent.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the discount rate (as a decimal). The discount is $\boldsymbol{D}=\mathbf{3 4 . 9 9}-\mathbf{2 7 . 9 9}=\mathbf{7 . 0 0}$.
Equation:
Discount $=$ discount rate of regular price

$$
\begin{aligned}
7 & =(x)(34.99) \\
\frac{7}{34.99} & =\frac{34.99}{34.99} x \\
0.20 & =x \\
x & =20 \%
\end{aligned}
$$

The discount rate is $\mathbf{2 0 \%}$.
Example 6:
Find the regular price of a mattress that sells for $\$ 469.99$ if the discount rate is $\mathbf{3 3 . 3 3 \%}$

## Solution:

Preamble:

Let $\boldsymbol{R}$ be the the regular price.
Then the discount is $\boldsymbol{D}=\mathbf{0 . 3 3 3 3 R}$ and the sales price is $R=S+0.3333 R, S=R-0.3333 R=0.6667 R$.

Equation:
Regular price $=$ Regular price
$0.6667 R=469.99$
$\frac{0.6667}{0.6667} R=\frac{469.99}{0.6667}=704.95$
The regular price is $\mathbf{\$ 7 0 4 . 9 5}$
Check: 704.95-0.3333(704.95) $=469.99$
Example 7:
A retailer buys a pair of boots for $\$ \mathbf{2 0 0}$. The markup is $\mathbf{1 0 \%}$ or $\$ \mathbf{2 0}$. The sales price is $\mathbf{2 0 0}+\mathbf{2 0}=\mathbf{2 2 0}$.
A year later, the previous sales price is now called the regular price and the boots are discounted at a discount rate of $\mathbf{1 0 \%}$.

Since the markup rate equals the discount rate, has the retailer made a profit, broken even, or lost money?

## Solution:

Preamble:
$D=(0.1)(220)=\$ 22$
Equation:
$S=R-D=220-22=198$.
The retailer lost $\mathbf{\$ 2}$.
Note that markup is computed on $\$ 200$ while discount is calculated on $\$ \mathbf{2 2 0}$.

### 14.4 Exercise 14

1. The Golden Nest buys a wedding ring at a cost of $\mathbf{\$ 1 , 3 9 5}$. The Markup rate is $\mathbf{6 5 \%}$. What is the markup and what is the sales price?
2. The Golden Nest buys a necklace at a cost of $\mathbf{\$ 2 3 5}$. It is on sale for $\mathbf{\$ 6 7 5}$. Find the markup rate (as a percent). Round to the nearest percent.
3. The Golden Nest sells a bracelet for $\mathbf{\$ 5 9 . 9 9}$. The markup rate is $\mathbf{3 5 \%}$. What was the cost of the bracelet? (Round to the nearest cent.)
4. A $\$ \mathbf{1}, \mathbf{2 5 0}$ (Regular price) computer has been sitting in at TechTech store for over a year. The merchant encourages you to buy the computer after applying a $\mathbf{3 3 \%}$ discount. How much are you
buying the computer for?
5. The regular price of a phone CallMe is $\mathbf{8 9 . 9 9}$ (regular price). It is discounted to sell for $\mathbf{\$ 6 7 . 9 9}$. What is the discount rate (in percent)? Round to the nearest percent.
6. Find the regular price of a TV set that sells for $\$ \mathbf{3 4 5}$ if the discount rate is $\mathbf{2 1 \%}$

## STOP!

1. The Golden Nest buys a wedding ring at a cost of $\mathbf{\$ 1}, \mathbf{3 9 5}$. The Markup rate is $\mathbf{6 5 \%}$. What is the markup and what is the sales price?

## Solution:

Pxeasnbe the sales price.
The cost is $\mathbf{\$ 1 , 3 9 5}$ and the markup rate is $\mathbf{6 5 \%}=\mathbf{0 . 6 5}$.
Equation:

$$
\begin{aligned}
\text { Markup } & =\text { percent(decimal) of Cost } \\
\boldsymbol{M} & =(\mathbf{0 . 6 5})(\mathbf{1}, \mathbf{3 9 5}) \\
& =\mathbf{9 0 6 . 7 5}
\end{aligned}
$$

The markup is $\$ 906.75$.
The sales price is $\mathbf{1 , 3 9 5}+\mathbf{9 0 6 . 7 5}=\$ 2,301.75$.
Note: The markup is $\mathbf{0 . 6 5}$ for every dollar of cost. Then the sales price is $\mathbf{1 . 0}+\mathbf{0 . 6 5}=\mathbf{1 . 6 5}$ dollars. We could have found the sales price more quickly by multiplying 1,395 by 1.65 .
Check: $1.65 \cdot 1,395=2301.75$
2. The Golden Nest buys a necklace at a cost of $\mathbf{\$ 2 3 5}$. It is on sale for $\mathbf{\$ 6 7 5}$. Find the markup rate (as a percent). Round to the nearest percent.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the markup rate (as a decimal).
The cost is $\$ \mathbf{2 3 5}$, the sales price is $\mathbf{\$ 6 7 5}$. The markup is $\mathbf{6 7 5} \mathbf{- 2 3 5}=\mathbf{\$ 4 4 0}$
Equation:
Markup $=$ percent(decimal) of Cost
$440=(x)(235)$
$\frac{440}{235}=\frac{235}{235} x$
$\frac{88}{47}=x$
$x=1.8723$
$x=187.23 \%$

The markup rate is $\mathbf{1 8 7 \%}$.
Check: $(\mathbf{1}+\mathbf{1 . 8 7}) \cdot \mathbf{2 3 5}=\mathbf{6 7 4 . 4 5}$ (the $\mathbf{5 5}$ cents discrepancy is the result of rounding.)
3. The Golden Nest sells a bracelet for $\mathbf{\$ 5 9 . 9 9}$. The markup rate is $\mathbf{3 5 \%}$. What was the cost of the bracelet? (Round to the nearest cent.)

## Solution:

Preamble:
Let $\boldsymbol{C}$ be the cost. Then the markup is $\boldsymbol{M}=\mathbf{0 . 3 5 C}$ and the sales price is $S=C+0.35 C=1.35 C$.

Equation:
Sales price $=$ Sales price

$$
1.35 C=59.99
$$

$$
\frac{1.35}{1.35} C=\frac{59.99}{1.35} x
$$

$$
C=44.44
$$

Check: $M=\mathbf{0 . 3 5 ( 4 4 . 4 4 )}=\$ 15.55$,
$S=44.44+15.55=59.99$.
4. A $\$ \mathbf{1}, \mathbf{2 5 0}$ (Regular price) computer has been sitting in at TechTech store for over a year. The merchant encourages you to buy the computer after applying a $\mathbf{3 3 \%}$ discount. How much are you buying the computer for?

## Solution:

Preamble:
Let $\boldsymbol{S}$ be the sales price. The discount is $\mathbf{3 3 \%}$ of the regular price.
Equation:
Discount $=$ discount rate of regular price

$$
\begin{aligned}
D & =(0.33)(1,250) \\
& =412.50
\end{aligned}
$$

The sales price $S=1,250-412.50=\$ 837.50$
Check: $\mathbf{1 , 2 5 0} * \mathbf{0 . 6 7}=\mathbf{8 3 7 . 5 0}$
5. The regular price of a phone CallMe is $\mathbf{8 9 . 9 9}$ (regular price). It is discounted to sell for $\mathbf{\$ 6 7 . 9 9}$. What is the discount rate (in percent)? Round to the nearest percent.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the discount rate (as a decimal). The discount is $\boldsymbol{D}=\mathbf{8 9 . 9 9 - 6 7 . 9 9}=\mathbf{2 2 . 0 0}$.
Equation:

```
Discount \(=\) discount rate of regular price
        \(22=(x)(89.99)\)
    \(\frac{22}{89.99}=\frac{89.99}{89.99} x\)
\(0.2445=x\)
    \(x=24.45 \%\)
```

The discount rate is $\mathbf{2 4 \%}$.
6. Find the regular price of a TV set that sells for $\$ \mathbf{3 4 5}$ if the discount rate is $\mathbf{2 1 \%}$

## Solution:

Preamble:
Let $\boldsymbol{R}$ be the regular price.
Then the discount is $\boldsymbol{D}=\mathbf{0 . 2 1 R}$ and the sales price is $\boldsymbol{R}=\boldsymbol{S}+\mathbf{0 . 2 1 R}, \boldsymbol{S}=\boldsymbol{R}-\mathbf{0 . 2 1 R}=\mathbf{0 . 7 9 R}$.
Equation:
Regular price $=$ Regular price

$$
\begin{aligned}
0.79 R & =345 \\
\frac{0.79}{0.79} R & =\frac{345}{0.79} \\
& =436.71
\end{aligned}
$$

The regular price is $\$ 436.71$
Check: $436.71-\mathbf{0 . 2 1}(436.71)=345$

## Chapter 15

## Solving Linear Equations. Investment Problems

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### 15.1 Youtube

http://www.youtube.com/playlist?list=PLAC117AE8B869758E\&feature=view_all
Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

Investment problems normally deal with compound interest. We need to understand simple interest first. We deal with simple interest in the following problems.

When you invest money you want to be rewarded. Your reward, called interest, depends on three quantities:

1) The more your investment (Principal $\boldsymbol{P}$ ) the more your reward.
2) The more money you get per $\mathbf{1 0 0}$ dollars invested (rate of interest, or just rate $\boldsymbol{r}$ ), the higher your reward.
3) The longer you invest your money (time $\boldsymbol{t}$ usually for a number of years or a fraction of a year). We assume here that the rate of interest $\boldsymbol{r}$ is in years.

Thus

$$
\mathrm{I}=\mathrm{Prt}
$$

### 15.2 Examples

Example 1:
Your son invests $\$ 20,000$ at $5.5 \%$ and your daughter invests $\$ 15,000$ at $3.5 \%$ simple interest for one year. What is the total interest earned?

## Solution:

Preamble:

|  | Principal | $\cdot$ | Rate | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Son | $\mathbf{2 0 , 0 0 0}$ | $\cdot$ | $\mathbf{0 . 0 5 5}$ | $=$ | $\mathbf{1 , 1 0 0}$ |
| Daughter | $\mathbf{1 5 , 0 0 0}$ | $\cdot$ | $\mathbf{0 . 0 3 5}$ | $=$ | $\mathbf{5 2 5}$ |

Equation:
Total Interest $=$ Interest from added to interest from daughter.
Total interest $=\mathbf{1 , 1 0 0}+\mathbf{5 2 5}=\mathbf{\$ 1 , 6 2 5}$
Example 2:
Your grandmother invests part of $\mathbf{\$ 4 , 5 0 0}$ at $\mathbf{8 \%}$ and the rest at $\mathbf{6 \%}$. Her annual interest income is $\mathbf{\$ 2 9 0}$. How much money did she invest at $8 \%$ ?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the amount invested at $\mathbf{8 \%}$ (like $\mathbf{\$ 1 , 0 0 0 ) . ~}$
Then $\mathbf{4 , 5 0 0}-\boldsymbol{x}$ is the amount invested at $\mathbf{6 \%}$
(like $4,500-1,000$ )

|  | Principal | $\cdot$ | Rate | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $8 \%$ account | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 8}$ | $=$ | $\mathbf{0 . 0 8 x}$ |
| $\mathbf{6 \%}$ account | $4,500-\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 6}$ | $=$ | $\mathbf{0 . 0 6 ( 4 , 5 0 0 - \boldsymbol { x } )}$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =290 \\
0.08 x+0.06(4,500-x) & =290 \\
0.08 x+270-0.06 x & =290 \\
0.02 x+270 & =290 \\
0.02 x+270-270 & =290-270 \\
0.02 x & =20 \\
\frac{0.02}{0.02} x & =\frac{20}{0.02} \\
x & =\frac{2000}{2} \\
x & =1,000
\end{aligned}
$$

$\mathbf{1 , 0 0 0}$ dollars was invested at $\mathbf{8 \%}$.
Example 3:
An entrepreneur borrowed some money at $\mathbf{4 \%}$ and $\$ 1,400$ more at $\mathbf{7 \%}$ annually. His interest is $\$ 768$ for two years. How much did the entrepreneur borrow at $\mathbf{7 \%}$ ?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the amount borrowed at $\mathbf{4 \%}$ (like $\mathbf{\$ 1 , 0 0 0 ) .}$
Then $\boldsymbol{x}+1,400$ is the amount borrowed at $\mathbf{7 \%}$
(like $1,000+1,400)$

|  | Principal | $\cdot$ | Rate | $\cdot$ | Time | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{4 \%}$ account | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 4}$ | $\cdot$ | 2 | $=$ | $(\mathbf{0 . 0 4 ) ( 2 ) \boldsymbol { x }}$ |
| $\mathbf{7 \%}$ account | $\boldsymbol{x}+\mathbf{1 , 4 0 0}$ | $\cdot$ | $\mathbf{0 . 0 7}$ | $\cdot$ | $\mathbf{2}$ | $=$ | $(\mathbf{0 . 0 7})(\mathbf{2})(\boldsymbol{x}+\mathbf{1}, \mathbf{4 0 0})$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =768 \\
(0.04)(2) x+(0.07)(2)(x+1,400) & =768 \\
(0.08) x+(0.14)(x+1,400) & =768 \\
(0.08) x+(0.14) x+(0.14)(1,400) & =768 \\
0.22 x+196 & =768 \\
0.22 x & =572 \\
\frac{0.22}{0.22} x & =\frac{572}{0.22} \\
x & =\frac{572}{0.22} \\
x & =2,600
\end{aligned}
$$

$2,600+1,400=4,000$ dollars was invested at $7 \%$.
Example 4:
A banker invests $\mathbf{3 0 \%}$ of a client's money at $\mathbf{5 \%}$ and the rest at $\mathbf{6 \%}$ annual interest rate. Both investments are for $\mathbf{3}$ months. The interest earned is $\mathbf{\$ 7 1 2 . 5 0}$. How much money did the client make available to the banker?

## Solution:

Preamble:
The rest of the money is $\mathbf{1 0 0 \%}-\mathbf{3 0 \%}=\mathbf{7 0 \%}$.
Let $\boldsymbol{x}$ be the client's money.
$0.30 x$ was invested at $\mathbf{5 \%}$ (like $\mathbf{3 0 \%}$ of $\$ 1,000=300$ ).
Then $\mathbf{0 . 7 0 \boldsymbol { x }}$ is the amount invested at $\mathbf{6 \%}$
(like $\mathbf{7 0 \%}$ of $\mathbf{1 , 0 0 0}=\mathbf{7 0 0}$ )
The time is $\mathbf{3}$ months which means
$\frac{3 \text { months }}{1} \cdot \frac{1 \text { year }}{12 \text { months }}=\frac{\mathbf{1}}{4}$ year $=\mathbf{0 . 2 5}$ year.

|  | Principal | $\cdot$ | Rate | $\cdot$ | Time | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{5 \%}$ account | $0.3 \boldsymbol{x}$ | $\cdot$ | 0.05 | $\cdot$ | 0.25 | $=$ | $(\mathbf{0 . 0 5})(\mathbf{0 . 2 5})(0.3) \boldsymbol{x}$ |
| $\mathbf{6 \%}$ account | $\mathbf{0 . 7 x}$ | $\cdot$ | $\mathbf{0 . 0 6}$ | $\cdot$ | $\mathbf{0 . 2 5}$ | $=$ | $(\mathbf{0 . 0 6 ) ( 0 . 2 5 ) ( 0 . 7 x )}$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =712.5 \\
(0.05)(0.25)(0.3) x+(0.06)(0.25)(0.7 x) & =712.5 \\
(0.0125)(0.3) x+(0.015)(0.7 x) & =712.5 \\
0.00375 x+0.01050 x & =712.5 \\
375 x+1,050 x & =71,250,000 \\
1,425 x & =71,250,000 \\
\frac{1,425}{1,425} x & =\frac{71,250,000}{1,425} \\
x & =\frac{71,250,000}{1,425} \\
x & =50,000
\end{aligned}
$$

The client trusted the banker with $\mathbf{\$ 5 0}, \mathbf{0 0 0}$.

### 15.3 Exercise 15

1. Armand invests $\$ \mathbf{1 0}, 000$ at $\mathbf{2 . 5 \%}$ and your Bonnie invests $\$ \mathbf{5}, \mathbf{0 0 0}$ at $\mathbf{4 \%}$ simple interest for one year. What is the total interest earned?
2. Carina invests part of $\mathbf{\$ 6}, \mathbf{0 0 0}$ at $\mathbf{9 \%}$ and the rest at $\mathbf{7 \%}$ simple annual interest rate. Her annual interest income is $\$ \mathbf{4 7 5}$. How much money did she invest at $\mathbf{9 \%}$ ?
3. An engineer borrowed some money at $\mathbf{6 \%}$ and $\$ \mathbf{2 , 5 0 0}$ more at $\mathbf{1 0 \%}$ simple annual interest rates. His interest is $\mathbf{\$ 1 , 4 5 0}$ for one year. How much did the engineer borrow at $\mathbf{6 \%}$ ?
4. A realtor invests $\mathbf{4 0 \%}$ of a client's money at $\mathbf{6 \%}$ and the rest at $\mathbf{1 1 \%}$ simple annual interest, both for 6 months. The interest earned is $\mathbf{\$ 1 , 0 6 8 . 7 5}$. How much money did the client make available to the realtor?

## STOP!

1. Armand invests $\$ \mathbf{1 0}, \mathbf{0 0 0}$ at $\mathbf{2 . 5 \%}$ and your Bonnie invests $\mathbf{\$ 5}, \mathbf{0 0 0}$ at $\mathbf{4 \%}$ simple interest for one year. What is the total interest earned?

## Solution:

Preamble:

|  | Principal | $\cdot$ | Rate | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Armand | $\mathbf{1 0 , 0 0 0}$ | $\cdot$ | $\mathbf{0 . 0 2 5}$ | $=$ | $\mathbf{2 5 0}$ |
| Bonnie | $\mathbf{5 , 0 0 0}$ | $\cdot$ | $\mathbf{0 . 0 4}$ | $=$ | $\mathbf{2 0 0}$ |

Equation:
Total Interest $=$ Interest from son added to interest from daughter.
Total interest $=\mathbf{2 5 0}+\mathbf{2 0 0}=\mathbf{\$ 4 5 0}$
2. Carina invests part of $\$ \mathbf{6}, \mathbf{0 0 0}$ at $\mathbf{9 \%}$ and the rest at $\mathbf{7 \%}$ simple annual interest rate. Her annual interest income is $\mathbf{\$ 4 7 5}$. How much money did she invest at $\mathbf{9 \%}$ ?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the amount invested at $\mathbf{9 \%}$ (like $\$ 1,000$ ).
Then $\mathbf{6 , 0 0 0}-\boldsymbol{x}$ is the amount invested at $\mathbf{7 \%}$
(like 6, $000-1,000$ )

|  | Principal | $\cdot$ | Rate | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{9 \%}$ account | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 9}$ | $=$ | $\mathbf{0 . 0 9 \boldsymbol { x }}$ |
| $\mathbf{7 \%}$ account | $\mathbf{6 , 0 0 0}-\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 7}$ | $=$ | $\mathbf{0 . 0 7}(\mathbf{6 , 0 0 0}-\boldsymbol{x})$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =475 \\
0.09 x+0.07(6,000-x) & =475 \\
0.09 x+0.07(6,000)-(0.07) x & =475 \\
0.09 x+420-0.07 x & =475 \\
0.02 x+420 & =475 \\
0.02 x+420-420 & =475-420 \\
-0.02 x & =-55 \\
\frac{-0.02}{-0.02} x & =\frac{-55}{-0.02} \\
x & =\frac{5,500}{2} \\
x & =2,750
\end{aligned}
$$

Carina invested 2, 750 at $\mathbf{9 \%}$.
3. An engineer borrowed some money at $\mathbf{6 \%}$ and $\mathbf{\$ 2 , 5 0 0}$ more at $\mathbf{1 0 \%}$ simple annual interest rates. His interest is $\$ \mathbf{1}, \mathbf{4 5 0}$ for one year. How much did the engineer borrow at $\mathbf{6 \%}$ ?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the amount borrowed at $\mathbf{6 \%}$ (like $\mathbf{\$ 1 , 0 0 0 )}$.
Then $\boldsymbol{x}+\mathbf{2 , 5 0 0}$ is the amount borrowed at $\mathbf{1 0 \%}$ (like $\mathbf{1 , 0 0 0}+\mathbf{1 , 5 0 0}$ )

|  | Principal | $\cdot$ | Rate | $\cdot$ | Time | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{6 \%}$ account | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{0 . 0 6}$ | $\cdot$ | $\mathbf{1}$ | $=$ | $(\mathbf{0 . 0 6 ) \boldsymbol { x }}$ |
| $\mathbf{1 0 \%}$ account | $\boldsymbol{x}+\mathbf{2 , 5 0 0}$ | $\cdot$ | $\mathbf{0 . 1}$ | $\cdot$ | $\mathbf{1}$ | $=$ | $\mathbf{( 0 . 1})(\boldsymbol{x}+\mathbf{2}, \mathbf{5 0 0})$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =1,450 \\
0.06 x+0.1(x+2,500) & =1,450 \\
0.06 x+0.1 x+250 & =1,450 \\
0.16 x+250 & =1,450 \\
0.16 x+250-250 & =1,450-250 \\
0.16 x & =1,200 \\
\frac{0.16}{0.16} x & =\frac{1,200}{0.16} \\
x & =\frac{1,200}{0.16} \\
x & =7,500
\end{aligned}
$$

The engineer borrowed $\mathbf{7 , 5 0 0}$ dollars at $\mathbf{6 \%}$.
4. A realtor invests $\mathbf{4 0 \%}$ of a client's money at $\mathbf{6 \%}$ and the rest at $\mathbf{1 1 \%}$ simple annual interest, both for 6 months. The interest earned is $\mathbf{\$ 1 , 0 6 8 . 7 5}$. How much money did the client make available to the realtor?

## Solution:

Preamble:
The rest of the money is $\mathbf{1 0 0 \%}-\mathbf{4 0 \%}=\mathbf{6 0 \%}$.
Let $\boldsymbol{x}$ be the client's money.
$0.40 x$ was invested at $\mathbf{6 \%}$ (like $\mathbf{4 0 \%}$ of $\$ 1,000=400$ ).
Then $\mathbf{0 . 6 0 \boldsymbol { x }}$ is the amount invested at $\mathbf{1 1 \%}$
(like $\mathbf{6 0 \%}$ of $\mathbf{1 , 0 0 0}=\mathbf{6 0 0}$ )
The time is $\mathbf{6}$ months which means
$\frac{6 \text { months }}{1} \cdot \frac{\mathbf{1} \text { year }}{\mathbf{1 2} \text { months }}=\frac{\mathbf{1}}{2}$ year $=\mathbf{0 . 5}$ year.

|  | Principal | $\cdot$ | Rate | $\cdot$ | Time | $=$ | Interest |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{6 \%}$ account | $\mathbf{0 . 4 \boldsymbol { x }}$ | $\cdot$ | $\mathbf{0 . 0 6}$ | $\cdot$ | $\mathbf{0 . 5}$ | $=$ | $(\mathbf{0 . 0 6 ) ( 0 . 5 ) ( \mathbf { 0 . 4 } ) \boldsymbol { x }}$ |
| $\mathbf{1 1 \%}$ account | $\mathbf{0 . 6 \boldsymbol { x }}$ | $\cdot$ | $\mathbf{0 . 1 1}$ | $\cdot$ | $\mathbf{0 . 5}$ | $=$ | $\mathbf{( 0 . 1 1 ) ( 0 . 5 ) ( 0 . 6 \boldsymbol { x } )}$ |

Equation:

$$
\begin{aligned}
\text { Interest } 1+\text { interest } 2 & =1,068.75 \\
(0.06)(0.5)(0.4) x+(0.11)(0.5)(0.6 x) & =1,068.75 \\
(0.03)(0.4) x+(0.055)(0.6 x) & =1,068.5 \\
0.012 x+0.033 x & =1,068.75 \\
0.045 x & =1,068.75 \\
\frac{0.045}{0.045} x & =\frac{1,068.75}{0.045} \\
x & =\frac{1,068.75}{0.045} \\
x & =23,750
\end{aligned}
$$

The client trusted the realtor with $\mathbf{\$ 2 3}, \mathbf{7 5 0}$.

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## Chapter 16

## Solving Linear Equations. Mixture Problems

(c) H. Feiner 2011

### 16.1 Youtube

http://www. youtube.com/playlist?list=PL9112194A0BAE1246\&feature=view_all

### 16.2 General concepts

Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

Mixture problems deal with (at least) two quantities (liquids, grains, nuts, ...) of different concentrations
of an ingredient (alcohol, salt, nuts at one price, ... ). We need to remember two basic principles:
When the contents of two buckets are poured into a third bucket,
(1) the amount of substance in bucket $\mathbf{3}$ is the sum of the amounts of substance in buckets $\mathbf{1}$ and $\mathbf{2}$.
(2) the ingredient in bucket $\mathbf{3}$ is the sum of the ingredients in buckets $\mathbf{1}$ and $\mathbf{2}$.


Ingredient 60\% alcohol
Ingredient 30\% alcohol
Ingredient 50\% alcohol
$\mathbf{0 . 6} \cdot \mathbf{2 0}=\mathbf{1 2}$ ounces alcohol
$\mathbf{0 . 3} \cdot \mathbf{1 0}=\mathbf{3}$ ounces alcohol

$$
0.5 \cdot \mathbf{3 0}=\mathbf{1 5} \text { ounces alcohol }
$$

12 ounces alcohol $+\mathbf{3}$ ounces alcohol $=15$ ounces alcohol $\quad$ (principle 2)

### 16.3 Examples

Example 1:
How much of a $\mathbf{5 0 \%}$ antifreeze solution must be added to $\mathbf{3 0}$ gallons of a $\mathbf{1 0 \%}$ solution to obtain a $\mathbf{4 0 \%}$ antifreeze solution?

## Solution:

Preamble:

|  |  | Quantity | $\cdot$ | $\%($ decimal $)$ | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{1 0 \%}$ solution | $\mathbf{3 0}$ gallons of sol. | $\cdot$ | $\mathbf{0 . 1}$ | $=$ | $\mathbf{3}$ gal. of antifreeze |
| $\mathbf{( 2 )}$ | $\mathbf{5 0 \%}$ solution | $\boldsymbol{x}$ gallons solution | $\cdot$ | $\mathbf{0 . 5}$ | $=$ | $\mathbf{0 . 5 \boldsymbol { x }}$ gal. of antifreeze |
| $\mathbf{( 3 )}$ | $\mathbf{4 0 \%}$ solution | $\mathbf{3 0 + \boldsymbol { x }}$ gal. solution | $\cdot$ | $\mathbf{0 . 4}$ | $=$ | $\mathbf{0 . 4 ( \boldsymbol { x } + \mathbf { 3 0 } ) \text { gal. antif. }}$ |

Equation:

Antifreeze in bucket $\mathbf{1}+$ antifreeze in bucket $\mathbf{2}=$ antifreeze in bucket $\mathbf{3}$

$$
\begin{aligned}
3+0.5 x & =0.4(x+30) \\
3+0.5 x & =0.4 x+12 \\
3+0.5 x-0.4 x & =0.4 x-0.4 x+12 \\
3+0.1 x & =12 \\
3-3+0.1 x & =12-3 \\
0.1 x & =9 \\
\frac{0.1}{0.1} x & =\frac{9}{0.1} \\
x & =\frac{90}{1} \Rightarrow x=90
\end{aligned}
$$

Mix $\mathbf{3 0}$ gallons of $\mathbf{1 0 \%}$ antifreeze with $\mathbf{9 0}$ gallons of $\mathbf{5 0 \%}$ antifreeze to get a $\mathbf{4 0 \%}$ antifreeze solution.
Example 2:
$\mathbf{5 0}$ gallons of a $\mathbf{2 0 \%}$ acid solution are mixed with $\mathbf{2 0}$ gallons of a $\mathbf{9 0 \%}$ acid solution. What is the acid concentration of the resulting mixture?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the concentration of the mixture (as a decimal).

|  |  | Quantity | $\cdot$ | $\%$ (decimal) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{2 0 \%}$ solution | $\mathbf{5 0}$ gal. solution | $\cdot$ | $\mathbf{0 . 2}$ | $=$ | $\mathbf{1 0}$ gallons of acid |
| $\mathbf{( 2 )}$ | $\mathbf{9 0 \%}$ solution | $\mathbf{2 0}$ gal. solution | $\cdot$ | $\mathbf{0 . 9}$ | $=$ | $\mathbf{0 . 9 ( \mathbf { 2 0 } ) = \mathbf { 1 8 } \text { gal. acid }}$ |
| $\mathbf{( 3 )}$ | $\mathbf{1 0 0 x}$ solution | $\mathbf{7 0}$ gal. solution | $\cdot$ | $\boldsymbol{x}$ | $=$ | $\mathbf{7 0 x}$ gal. acid |

$\boldsymbol{x}$ is a decimal. That's why it is multiplied by 100.
Equation:
Acid in bucket $\mathbf{1}+$ acid in bucket $\mathbf{2}=$ acid in bucket $\mathbf{3}$

$$
\begin{aligned}
10+18 & =70 x \\
28 & =70 x \\
\frac{28}{70} & =\frac{70 x}{70} \\
\frac{4}{10} & =x \\
x & =\frac{40}{100}
\end{aligned}
$$

The concentration of the acid mixture is $\mathbf{4 0 \%}$.

## Example 3:

How much of pure ( $\mathbf{1 0 0 \%}$ ) orange juice must be added to $\mathbf{1 0}$ gallons of a $\mathbf{1 5 \%}$ orange juice solution to obtain a $\mathbf{5 0 \%}$ orange juice solution?

## Solution:

Preamble:

|  |  | Quantity | $\cdot$ | $\%$ (decimal) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(\mathbf{1})$ | $\mathbf{1 5 \%}$ solution | $\mathbf{1 0}$ gal.solution | $\cdot$ | $\mathbf{0 . 1 5}$ | $=$ | $\mathbf{1 . 5}$ gal.orange juice |
| $\mathbf{( 2 )}$ | $\mathbf{1 0 0 \%}$ orange juice | $\boldsymbol{x}$ gallons o. j. | $\cdot$ | $\mathbf{1 . 0}$ | $=$ | $\boldsymbol{x}$ gal.orange juice |
| $\mathbf{( 3 )}$ | $\mathbf{5 0 \%}$ solution | $\mathbf{1 0}+\boldsymbol{x}$ gal. sol. | $\cdot$ | $\mathbf{0 . 5}$ | $=$ | $\mathbf{0 . 5 ( \boldsymbol { x } + \mathbf { 1 0 } ) \text { gal. } \text { o. } \mathrm { j } .}$ |

## Equation:

Orange juice in bucket $\mathbf{1}+$ orange juice in bucket $\mathbf{2}=$ orange juice in bucket $\mathbf{3}$

$$
\begin{aligned}
1.5+x & =0.5(x+10) \\
1.5+x & =0.5 x+5 \\
1.5+x-0.5 x & =0.5 x-0.5 x+5 \\
1.5+0.5 x & =5 \\
1.5-1.5+0.5 x & =5-1.5 \\
0.5 x & =3.5 \\
\frac{0.5}{0.5} x & =\frac{3.5}{0.5} \\
x & =\frac{35}{5}=7
\end{aligned}
$$

Mix $\mathbf{1 0}$ gallons of $\mathbf{1 5 \%}$ orange juice solution with $\mathbf{7}$ gallons of pure orange juice to get a $\mathbf{5 0 \%}$ orange juice solution.

Example 4:
$\mathbf{1 0}$ gallons of a $\mathbf{4 5 \%}$ peroxide solution are mixed with how many gallons of pure water (no peroxide) to obtain $\mathbf{1 5 \%}$ peroxide solution?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of gallons of pure water.

|  |  | Quantity | $\cdot$ | $\%$ (decimal) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{4 5 \%}$ solution | $\mathbf{1 0}$ gallons of solution | $\cdot$ | $\mathbf{0 . 4 5}$ | $=$ | $\mathbf{4 . 5}$ gal.peroxide |
| $\mathbf{( 2 )}$ | $\mathbf{0 \%}$ pure water | $\boldsymbol{x}$ gal. solution | $\cdot$ | $\mathbf{0}$ | $=$ | $\mathbf{0}$ gallons of peroxide |
| $\mathbf{( 3 )}$ | $\mathbf{1 5 \%}$ solution | $\boldsymbol{x}+\mathbf{1 0}$ gal. solution | $\cdot$ | $\boldsymbol{x}+\mathbf{1 0}$ | $=$ | $\mathbf{1 5}(\boldsymbol{x}+\mathbf{1 0})$ gal.peroxide |

Equation:

Peroxide in bucket $\mathbf{1}+$ peroxide in bucket $\mathbf{2}=$ peroxide in bucket $\mathbf{3}$

$$
\begin{aligned}
4.5+0 & =0.15(x+10) \\
4.5 & =0.15 x+1.5 \\
4.5-1.5 & =0.15 x+1.5-1.5 \\
3 & =0.15 x \\
\frac{3}{0.15} & =\frac{0.15}{0.15} x \\
\frac{300}{15} & =x \\
x & =20
\end{aligned}
$$

Add $\mathbf{2 0}$ gallons pure water to 10 gallons of $\mathbf{4 5 \%}$ peroxide solution to distill it to $\mathbf{1 5 \%}$ peroxide solution.
Example 5:
A grocer mixes $\mathbf{2 0}$ pounds of peanuts selling for $\frac{\mathbf{2} \text { dollars }}{\text { pound }}$ with $\mathbf{5}$ pounds of almonds selling for $\frac{\boldsymbol{7} \text { dollars }}{\text { pound }}$. What is the price of the mixture in $\frac{\text { dollars }}{\text { pound }} ?$

## Solution:

Preamble:

|  |  | Quantity | $\cdot$ | Price | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(\mathbf{1})$ | peanuts | $\mathbf{2 0}$ pounds | $\cdot$ | $\frac{\mathbf{2} \text { dollars }}{\text { pound }}$ | $=$ | $\mathbf{4 0}$ dollars |
| $(\mathbf{2})$ | Almonds | $\mathbf{5}$ pounds | $\cdot$ | $\frac{\mathbf{7} \text { dollars }}{\text { pound }}$ | $=$ | $\mathbf{3 5}$ dollars |
| $(\mathbf{3})$ | Mixture | $\mathbf{2 5}$ pounds | $\cdot$ | $\boldsymbol{x}$ | $=$ | $\mathbf{2 5 x}$ dollars |

Equation:
Price of nuts bucket $\mathbf{1}+$ price of in bucket $\mathbf{2}=$ price of nuts in bucket $\mathbf{3}$

$$
\begin{aligned}
40+35 & =25 x \\
75 & =25 x \\
\frac{75}{25} & =\frac{25}{25} x \\
x & =3
\end{aligned}
$$

The nuts in the mixture sell for $\frac{\mathbf{3} \text { dollars }}{\text { pound }}$

### 16.4 Exercise 16

1. How much of a $\mathbf{4 0 \%}$ sulfuric acid solution must be added to $\mathbf{2 0 0}$ gallons of a $\mathbf{8 0 \%}$ solution to obtain a $65 \%$ solution?
2. $\mathbf{4 0}$ grams of $\mathbf{3 5 \%}$ silver are mixed with $\mathbf{2 6 0}$ grams of $\mathbf{7 0 \%}$ silver. What is the concentration of the resulting mixture?
3. How much of pure ( $\mathbf{1 0 0 \%}$ ) gold must be added to $\mathbf{1 7 5}$ grams of $\mathbf{2 0 \%}$ gold to obtain a $\mathbf{6 0 \%}$ gold mixture?
4. $\mathbf{1 0 0}$ gallons of a $\mathbf{2 2 \%}$ alcohol solution are mixed with how many gallons of pure water (no alcohol) to obtain $\mathbf{8 \%}$ alcohol solution?
5. A horse rancher mixes $\mathbf{1 5}$ metric tons of hay selling for $\frac{\mathbf{1 2 0} \text { dollars }}{\text { ton }}$ with $\mathbf{3}$ tons of grain selling for $\frac{\mathbf{3 0 0} \text { dollars }}{\text { ton }}$. What is the price of the mixture in $\frac{\text { dollars }}{\text { ton }} ?$

## STOP!

1. How much of a $\mathbf{4 0 \%}$ sulfuric acid solution must be added to $\mathbf{2 0 0}$ gallons of a $\mathbf{8 0 \%}$ solution to obtain a $\mathbf{6 5 \%}$ solution?

## Solution:

Peeamble the quantity of $\mathbf{4 0 \%}$ sulfuric acid needed.

|  |  | Quantity | $\cdot$ | $\%($ dec. $)$ | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{4 0 \%}$ sol. | $\boldsymbol{x}$ gallons of sol. | $\cdot$ | $\mathbf{0 . 4}$ | $=$ | $\mathbf{0 . 4 \boldsymbol { x }}$ gallons of acid |
| $\mathbf{( 2 )}$ | $\mathbf{8 0 \%}$ sol. | $\mathbf{2 0 0}$ gallons sol. | $\cdot$ | $\mathbf{0 . 8}$ | $=$ | $\mathbf{1 6 0}$ gallons of acid |
| $\mathbf{( 3 )}$ | $\mathbf{6 5 \%}$ sol. | $\mathbf{2 0 0}+\boldsymbol{x}$ gal. sol. | $\cdot$ | $\mathbf{0 . 6 5}$ | $=$ | $\mathbf{0 . 6 5 ( \boldsymbol { x } + \mathbf { 2 0 0 } ) \text { gal. acid }}$ |

Equation:
Acid in bucket $\mathbf{1}+$ acid in bucket $\mathbf{2}=$ acid in bucket $\mathbf{3}$

$$
\begin{aligned}
0.4 x+160 & =0.65(x+200) \\
0.4 x+160 & =0.65 x+130 \\
0.4 x-0.65 x+160 & =0.65 x-0.65 x+130 \\
-0.25+160 & =130 \\
-0.25+160-160 & =130-160 \\
-0.25 & =-30 \\
\frac{-0.25}{-0.25} x & =\frac{-30}{-0.25} \\
x & =\frac{3000}{25} \\
x & =120
\end{aligned}
$$

Mix $\mathbf{1 2 0}$ gallons of $\mathbf{4 0 \%}$ sulfuric acid with $\mathbf{2 0 0}$ gallons of $\mathbf{8 0 \%}$ acid to get a $\mathbf{6 5 \%}$ acid solution.
2. $\mathbf{4 0}$ grams of $\mathbf{3 5 \%}$ silver are mixed with $\mathbf{2 6 0}$ grams of $\mathbf{7 0 \%}$ silver. What is the concentration of the resulting mixture?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the concentration of the mixture (as a decimal).

|  |  | Quantity | $\cdot$ | $\%$ (dec.) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{3 5 \%}$ silver | $\mathbf{4 0}$ grams | $\cdot$ | $\mathbf{0 . 3 5}$ | $=$ | $\mathbf{1 4}$ grams of silver |
| $\mathbf{( 2 )}$ | $\mathbf{7 0 \%}$ silver | $\mathbf{2 6 0}$ grams | $\cdot$ | $\mathbf{0 . 7}$ | $=$ | 182 grams of silver |
| $\mathbf{( 3 )}$ | $\mathbf{1 0 0 x} \boldsymbol{x}$ silver | $\mathbf{3 0 0}$ g of silver | $\cdot$ | $\boldsymbol{x}$ | $=$ | $\mathbf{3 0 0 \boldsymbol { x }}$ grams of acid |

Note $100 \boldsymbol{x}$ because $\boldsymbol{x}$ is a decimal.
Equation:
Silver in bucket $\mathbf{1}+$ silver in bucket $\mathbf{2}=$ silver in bucket $\mathbf{3}$

$$
\begin{aligned}
14+182 & =300 x \\
196 & =300 x \\
\frac{196}{300} & =\frac{300}{300} x \\
x & =\frac{196}{300} \\
x & =0.65 \frac{1}{3}
\end{aligned}
$$

The concentration of the silver mixture is $\mathbf{6 5} \frac{\mathbf{1}}{\mathbf{3}} \%$.
3. How much of pure ( $\mathbf{1 0 0 \%}$ ) gold must be added to $\mathbf{1 7 5}$ grams of $\mathbf{2 0 \%}$ gold to obtain a $\mathbf{6 0 \%}$ gold mixture?
Solution:
Preamble:

|  |  | Quantity | $\cdot$ | $\%$ (dec.) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathbf{( 1 )}$ | $\mathbf{1 0 0 \%}$ gold | $\boldsymbol{x}$ grams of gold | $\cdot$ | $\mathbf{1 . 0 0}$ | $=$ | $\boldsymbol{x}$ grams of gold |
| $(\mathbf{2})$ | $\mathbf{2 0 \%}$ gold | $\mathbf{1 7 5}$ grams | $\cdot$ | $\mathbf{0 . 2}$ | $=$ | $\mathbf{3 5}$ grams |
| $\mathbf{( 3 )}$ | $\mathbf{6 0 \%}$ gold | $\mathbf{1 7 5}+\boldsymbol{x}$ g of gold | $\cdot$ | $\mathbf{0 . 6}$ | $=$ | $\mathbf{0 . 6 ( \boldsymbol { x } + \mathbf { 1 7 5 } ) \text { g. gold }}$ |

Equation:

Orange juice in bucket $\mathbf{1}+$ orange juice in bucket $\mathbf{2}=$ orange juice in bucket $\mathbf{3}$

$$
\begin{aligned}
x+35 & =0.6(x+175) \\
x+35 & =0.6 x+105 \\
x+35-35 & =0.6 x+105-35 \\
x & =0.6 x+70 \\
x-0.6 x & =0.6 x-0.6 x+70 \\
0.4 x & =70 \\
\frac{0.4}{0.4} x & =\frac{70}{0.4} \\
x & =\frac{700}{4} \\
x & =175
\end{aligned}
$$

Mix 175 grams of pure ( $\mathbf{1 0 0 \%}$ ) gold with $\mathbf{1 7 5}$ grams of $\mathbf{2 0 \%}$ gold to get $\mathbf{6 0 \%}$ gold.
4. 100 gallons of a $\mathbf{2 2 \%}$ alcohol solution are mixed with how many gallons of pure water (no alcohol) to obtain $\mathbf{8 \%}$ alcohol solution?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of gallons of pure water.

|  |  | Quantity | $\cdot$ | $\%$ (dec.) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(\mathbf{1})$ | $\mathbf{2 2 \%}$ sol. | $\mathbf{1 0 0}$ gal. sol. | $\cdot$ | $\mathbf{0 . 2 2}$ | $=$ | $\mathbf{2 2}$ gal. alcohol |
| $(\mathbf{2})$ | $\mathbf{0 \%} \mathbf{H}_{\mathbf{2}} \mathbf{O}$ | $\boldsymbol{x}$ gal. water | $\cdot$ | $\mathbf{0}$ | $=$ | $\mathbf{0}$ gal. of alcohol |
| $\mathbf{( 3 )}$ | $\mathbf{8 \%}$ sol. | $\boldsymbol{x}+\mathbf{1 0 0}$ gal. sol. | $\cdot$ | $\mathbf{0 . 0 8}$ | $=$ | $\mathbf{0 8 ( \boldsymbol { 0 8 } + \mathbf { 1 0 0 } ) \text { gal. al. }}$ |

Equation:
Peroxide in bucket $\mathbf{1}+$ peroxide in bucket $\mathbf{2}=$ peroxide in bucket $\mathbf{3}$

$$
\begin{aligned}
22+0 & =0.08(x+100) \\
22 & =0.08 x+8 \\
22-8 & =0.08 x+8-8 \\
14 & =0.08 x \\
\frac{14}{0.08} & =\frac{0.08}{0.08} x \\
\frac{1400}{8} & =x \\
x & =175
\end{aligned}
$$

Add $\mathbf{1 7 5}$ gallons of pure water to $\mathbf{1 0 0}$ gallons of a $\mathbf{2 2 \%}$ alcohol solution to distill it to a $\mathbf{8 \%}$ alcohol solution.
5. A horse rancher mixes $\mathbf{1 5}$ metric tons of hay selling for $\frac{\mathbf{1 2 0} \text { dollars }}{\text { ton }}$ with $\mathbf{3}$ tons of grain selling for $\frac{\mathbf{3 0 0} \text { dollars }}{\text { ton }}$. What is the price of the mixture in $\frac{\text { dollars }}{\text { ton }}$ ?

## Solution:

Preamble:

|  |  | Quantity | $\cdot$ | Percent (decimal) | $=$ | Ingredient |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(\mathbf{1})$ | Hay | $\mathbf{1 5}$ tons | $\cdot$ | $\frac{\mathbf{1 2 0} \text { dollars }}{\text { ton }}$ | $=$ | $\mathbf{1}, \mathbf{8 0 0}$ dollars |
| $(\mathbf{2})$ | Grain | $\mathbf{3}$ tons | $\cdot$ | $\frac{\mathbf{3 0 0} \text { dollars }}{\text { ton }}$ | $=$ | $\mathbf{9 0 0}$ dollars |
| $(\mathbf{3})$ | Feed | $\mathbf{1 8}$ tons | $\cdot$ | $\frac{\boldsymbol{x} \text { dollars }}{\text { pound }}$ | $=$ | $\mathbf{1 8 x}$ dollars |

Equation:
Price of hay + price of grain $=$ price of feed

$$
\begin{aligned}
1,800+900 & =18 x \\
2,700 & =18 x \\
\frac{2,700}{18} & =\frac{18}{18} x \\
x & =150
\end{aligned}
$$

The feed sells for $\frac{\mathbf{1 5 0} \text { dollars }}{\text { ton }}$

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## Chapter 17

## Solving Linear Equations. Speed(Rate)-Distance-Time Problems

### 17.1 YouTube

https://www.youtube.com/playlist?list=PL09C56B4F258C43A0\&feature=view_all

### 17.2 Basics

Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

What is the legal speed limit on California freeways?

The speed (Rate equals Distance over Time) is

$$
\boldsymbol{R}=\frac{\boldsymbol{D} \text { miles }}{\boldsymbol{T} \text { hours }}
$$

Since

$$
\begin{align*}
\boldsymbol{R} & =\frac{\boldsymbol{D}}{\boldsymbol{T}} & (\mathrm{I}) \\
\boldsymbol{R T} & =\frac{\boldsymbol{D} \boldsymbol{T}}{\boldsymbol{T}} & \text { Multiply by } \boldsymbol{T} \text { ime } \\
\boldsymbol{R T} & =\boldsymbol{D} & \text { (II) }  \tag{II}\\
\frac{\boldsymbol{R} \boldsymbol{T}}{\boldsymbol{R}} & =\frac{\boldsymbol{D}}{\boldsymbol{R}} & \text { Divide by Rate } \\
\boldsymbol{T} & =\frac{\boldsymbol{D}}{\boldsymbol{R}} & \text { (III) } \tag{III}
\end{align*}
$$

The use of units cannot be overemphasized.

$$
\begin{align*}
& \boldsymbol{R}=\frac{\boldsymbol{D}}{\boldsymbol{T}} \\
& \frac{\text { miles }}{\text { hour }}=\frac{\text { miles }}{\text { hour }} \\
& \frac{\text { miles }}{\text { hour }} \cdot \frac{\boldsymbol{R T}=\boldsymbol{D}}{\boldsymbol{1}}=\text { miles } \\
& \boldsymbol{T}=\frac{\boldsymbol{D}}{\boldsymbol{R}} \\
& \text { hours }=\frac{\text { miles }}{\mathbf{1}} \div \frac{\text { miles }}{\text { hour }}=\frac{\text { miles }}{\mathbf{1}} \cdot \frac{\text { hour }}{\text { miles }} \tag{III}
\end{align*}
$$

It is usually helpful to sketch a diagram summarizing the problem.

### 17.3 Examples

Example 1:
Two cars leave a city at the same time from the same point and travel in opposite directions on a straight road. One car moves at $\mathbf{5 5} \mathrm{mph}$ the other's speed is $\mathbf{4 0} \mathrm{mph}$. How far apart are the cars after $\mathbf{3}$ hours?

## Solution:

Preamble:

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Car 1 | $\mathbf{4 0}$ | $\cdot$ | $\mathbf{3}$ | $=$ | $\mathbf{1 2 0}$ |
| Car 2 | $\mathbf{5 5}$ | $\cdot$ | $\mathbf{3}$ | $=$ | $\mathbf{1 6 5}$ |

$\underbrace{165 \text { miles }(\operatorname{car} 2)} \underbrace{120 \text { miles }(\operatorname{car} 1)}$
Equation:
Distance between cars $=$ car 1 distance + car 2 distance $=165+\mathbf{1 2 0}=\mathbf{2 8 5}$ miles
The distance between the cars is $\mathbf{2 8 5}$ miles.
Example 2:
Two cars leave a city at the same time from the same point and travel in the same direction on a straight road. One car moves at $\mathbf{5 5} \mathrm{mph}$ the other's speed is $\mathbf{4 0} \mathrm{mph}$. How far apart are the cars after $\mathbf{3}$ hours?

## Solution:

Preamble:

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Car 1 | $\mathbf{4 0}$ | $\cdot$ | $\mathbf{3}$ | $=$ | $\mathbf{1 2 0}$ |
| Car 2 | $\mathbf{5 5}$ | $\cdot$ | $\mathbf{3}$ | $=$ | $\mathbf{1 6 5}$ |



Equation:
$\overbrace{165 \text { miles }(\text { car } 2)}$
Distance between cars $=$ car 2 distance - car 1 distance $=165-120$ $=45$ miles

The distance between cars is $\mathbf{4 5}$ miles.
Example 3:
Two motor cyclists leave a city at the same time from the same point and travel in opposite directions on a straight road. One bike moves at 5 mph faster than the other. What is the speed of the slower bike if the riders are $\mathbf{1 6 2 . 5}$ miles apart after $\mathbf{2}$ hours $\mathbf{3 0}$ minutes?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the speed of the slower bike. $\mathbf{3 0}$ minutes is half an hour. The bikes travel for $\mathbf{2 . 5}$ hours.

Equation:

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :--- | :---: | :---: | :---: |
| Bike 1 | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{2 . 5}$ | $=$ | $\mathbf{2 . 5 x}$ |
| Bike $\mathbf{2}$ | $\boldsymbol{x}+\mathbf{5}$ | $\cdot$ | $\mathbf{2 . 5}$ | $=$ | $\mathbf{2 . 5} \boldsymbol{x}+\mathbf{5})$ |



Distance between cars $=$ Car 1 distance + Car $\mathbf{2}$ distance

$$
\begin{aligned}
2.5 x+2.5(x+5) & =162.5 \\
2.5 x+2.5 x+12.5 & =162.5 \\
5 x+12.5-12.5 & =162.5-12.5 \\
5 x & =150 \\
\frac{5}{5} x & =\frac{150}{5} \\
x & =30
\end{aligned}
$$

The slower bike travels at $\mathbf{3 0} \mathbf{m p h}$.
Example 4:
John drove a truck at 55 mph . He left at $8: 00$ A.M. At $9: 00$ A.M Joan discovers his medication and wallet. She hopped into her Ford Mustang and drove the same route at $\mathbf{6 5} \mathrm{mph}$. At what time did she catch up with John?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the time in hours it took Joan to catch John. John had traveled $\boldsymbol{x}+\mathbf{1}$ hours.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| John | $\mathbf{5 5}$ | $\cdot$ | $\boldsymbol{x}+\mathbf{1}$ | $=$ | $\mathbf{5 5 ( \boldsymbol { x } + 1 )}$ |
| Joan | $\mathbf{6 5}$ | $\cdot$ | $\boldsymbol{x}$ | $=$ | $\mathbf{6 5 x}$ |



Distance traveled by the Mustang $=$ distance traveled by the truck

$$
\begin{aligned}
65 x & =55(x+1) \\
65 x & =55 x+55 \\
65 x-55 x & =55 x-55 x+55 \\
10 x & =55 \\
\frac{10}{10} x & =\frac{55}{10} \\
x & =5.5
\end{aligned}
$$

Joan took 5 hours 30 minutes to catch John's truck at $9: \mathbf{0 0 + 5}: \mathbf{3 0}=\mathbf{1 4}: \mathbf{3 0}$ or $2: 30$ P.M.

Example 5:
A plane flies from LAX to JFK at $\mathbf{5 5 0} \mathrm{mph}$ and returns at $\mathbf{4 5 0} \mathrm{mph}$. Find the distance between the two airports if the total travel time is $\mathbf{1 0}$ hours.

Preamble:
Let $\boldsymbol{x}$ be the distance between LAX and JFK.
Remember that $\boldsymbol{T}=\frac{\boldsymbol{D}}{\boldsymbol{R}}$.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| LAX to JFK | $\mathbf{5 5 0}$ | $\cdot$ | $\frac{\boldsymbol{x}}{\mathbf{5 5 0}}$ | $=$ | $\boldsymbol{x}$ |
| JFK to LAX | $\mathbf{4 5 0}$ | $\cdot$ | $\frac{\boldsymbol{x}}{\mathbf{4 5 0}}$ | $=$ | $\boldsymbol{x}$ |

## Equation:

Time from LAX to JFK + time from JFK to LAX $=$ total time

$$
\begin{aligned}
\frac{x}{550}+\frac{x}{450} & =10 \\
\frac{x}{550} \cdot \frac{9}{9}+\frac{x}{450} \cdot \frac{11}{11} & =10 \\
\frac{9 x+11 x}{550 \cdot 9} & =10 \\
\frac{20 x \cdot 550 \cdot 9}{550 \cdot 9} & =10 \cdot 550 \cdot 9 \\
20 x & =5,500 \cdot 9 \\
\frac{20}{20} x & =\frac{5,500 \cdot 9}{20} \\
x & =2,475
\end{aligned}
$$

The distance between the two airports is $\mathbf{2 , 4 7 5}$ miles.
Note:
Students frequently find it easier to choose a different unknown in this problem.
Let $\boldsymbol{t}$ be the time it takes from LAX to JFK. Then the time from JFK to LAX is $\mathbf{1 0} \boldsymbol{-} \boldsymbol{t}$ hours.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| LAX to JFK | $\mathbf{5 5 0}$ | $\cdot$ | $\boldsymbol{t}$ | $=$ | $\mathbf{5 5 0} \boldsymbol{t}$ |
| JFK to LAX | $\mathbf{4 5 0}$ | $\cdot$ | $\mathbf{1 0}-\boldsymbol{t}$ | $=$ | $\mathbf{4 5 0}(\mathbf{1 0}-\boldsymbol{t})$ |

Equation:
Time from LAX to JFK $=$ time from JFK to LAX

$$
\begin{aligned}
550 t & =450(10-t) \\
550 t & =4,500-450 t \\
550 t+450 t & =4,500-450 t+450 t \\
1,000 t & =4,500 \\
\frac{1,000}{1,000} t & =\frac{4,500}{1,000} \\
t & =4.5
\end{aligned}
$$

The distance from LAX to JFK is $\mathbf{4 . 5 ( 5 5 0 )}=\mathbf{2 , 4 7 5}$ miles.
Example 6:
The speed of light is $\mathbf{1 8 6}, \mathbf{2 8 2} \frac{\text { miles }}{\text { second }}$ and the speed of sound $\mathbf{1 , 1 2 7} \frac{\mathrm{ft}}{\text { second }}$. LA is about $\mathbf{1 2 0}$ miles from San Diego.
There are $\mathbf{5 ,} \mathbf{2 8 0} \mathrm{ft}$ in a mile.
A bolt of lightning strikes somewhere in LA. Assume the bolt is seen and the thunder is heard in San Diego. How long after the lightning is seen will the thunder be heard in San Diego?

## Solution:

Preamble:
Sound travels at $\frac{\mathbf{1 , 1 2 7} \mathrm{ft}}{\text { second }}=\frac{\mathbf{1 , 1 2 7} \mathrm{ft}}{\text { second }} \cdot \frac{\text { mile }}{\mathbf{5 , 2 8 0} \mathrm{ft}}=\frac{\mathbf{0 . 2 1 3 4} \mathrm{miles}}{\text { second }}$

|  | Time | $=$ | Distance | $\div$ | Rate |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Lightning | $\boldsymbol{T}_{\boldsymbol{L}}$ | $=\frac{\mathbf{1 2 0} \text { miles }}{\mathbf{1}}$ | $\div$ | $\frac{\mathbf{1 8 6 , 2 8 2} \text { miles }}{\text { second }}$ |  |
| Sound | $\boldsymbol{T}_{\boldsymbol{S}}$ | $=$ | $\frac{\mathbf{1 2 0}}{\mathbf{1}}$ | $\div$ | $\frac{\mathbf{0 . 2 1 3 4} \text { miles }}{\text { second }}$ |

Equation:
$\boldsymbol{T}_{\boldsymbol{L}}=\frac{\mathbf{1 2 0} \text { miles }}{1} \cdot \frac{\text { second }}{\mathbf{1 8 6}, 282 \text { miles }}=6.44253 \times \mathbf{1 0}^{-4} \approx \mathbf{0 . 0 0 0 6}$ seconds.
$\boldsymbol{T}_{S}=\frac{\mathbf{1 2 0} \text { miles }}{\mathbf{1}} \cdot \frac{\text { second }}{\mathbf{0 . 2 1 3 4} \text { miles }}=\mathbf{5 6 2 . 3}$ seconds.
If it could be heard, thunder would be delayed by
$\mathbf{5 6 2}$ seconds $=\frac{\mathbf{5 6 2} \text { seconds }}{\mathbf{1}} \cdot \frac{\mathbf{1} \text { minute }}{\mathbf{6 0} \text { seconds }}=\mathbf{9}$ minutes 22 seconds.

### 17.4 Exercise 17

1. Two buses leave a city at the same time from the same point and travel in opposite directions on a straight road. One bus moves at $\mathbf{6 5} \mathrm{mph}$ the other's speed is $\mathbf{6 0} \mathrm{mph}$. How far apart are the buses after 5 hours?
2. Two buses leave a city at the same time from the same point and travel in the same direction on a straight road. One bus moves at $\mathbf{6 5} \mathrm{mph}$ the other's speed is $\mathbf{6 0} \mathrm{mph}$. How far apart are the buses after 5 hours?
3. A passenger and a freight train leave a city at the same time from the same point and travel in opposite directions on a straight road. The freight train moves at 40 mph slower than the passenger train. What is the speed of the freight train if the trains are $\mathbf{2 6 0}$ miles apart after $\mathbf{3}$ hours $\mathbf{1 5}$ minutes?
4. Abe rode a Moped at 35 mph . He left at 10 : $\mathbf{0 0}$ A.M. At noon Bill discovered Abe's driver's license and wallet at home. Bill rushed into his Bronco and drove the same route as Abe at $\mathbf{6 5} \mathrm{mph}$. At what time did Bill catch up with Abe?
5. A plane flies from LAX to ORD at $\mathbf{5 0 0} \mathbf{~ m p h}$ and returns at $\mathbf{3 5 0} \mathbf{~ m p h}$. Find the distance between the two airports if the total travel time is $\mathbf{8 . 5}$ hours.
6. The speed of light is $\mathbf{1 8 6}, \mathbf{2 8 2} \frac{\text { miles }}{\text { second }}$ and the speed of sound $\mathbf{1 , 1 2 7} \frac{\mathrm{ft}}{\text { second }}$. Santa Monica is about 20 miles from Long Beach.
There are $\mathbf{5 , 2 8 0} \mathrm{ft}$ in a mile.
A bolt of lightning strikes somewhere in Santa Monica. Assume the bolt is seen and the thunder is heard in Long Beach.
How long after the lightning is seen will the thunder be heard in Long Beach?

## STOP!

1. Two buses leave a city at the same time from the same point and travel in opposite directions on a straight road. One bus moves at $\mathbf{6 5} \mathrm{mph}$ the other's speed is $\mathbf{6 0} \mathrm{mph}$. How far apart are the cars after 5 hours?

## Solution:

Preamble:

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Bus 1 | $\mathbf{6 0}$ | $\cdot$ | 5 | $=$ | $\mathbf{3 0 0}$ |
| Bus 2 | $\mathbf{6 5}$ | $\cdot$ | 5 | $=$ | $\mathbf{3 2 5}$ |



Equation:
Distance between buses $=$ bus $\mathbf{1}$ distance + bus $\mathbf{2}$ distance $=\mathbf{3 2 5}+\mathbf{3 0 0}$

$$
=625 \text { miles }
$$

The distance between the buses is $\mathbf{6 2 5}$ miles.
2. Two buses leave a city at the same time from the same point and travel in the same direction on a straight road. One bus moves at $\mathbf{6 5} \mathrm{mph}$ the other's speed is $\mathbf{6 0} \mathrm{mph}$. How far apart are the cars after 5 hours?

## Solution:

Preamble:

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Bus 1 | $\mathbf{6 0}$ | $\cdot$ | $\mathbf{5}$ | $=$ | $\mathbf{3 0 0}$ |
| Bus 2 | $\mathbf{6 5}$ | $\cdot$ | $\mathbf{5}$ | $=$ | $\mathbf{3 2 5}$ |



Equation:
Distance between buses $=$ bus $\mathbf{2}$ distance - bus $\mathbf{1}$ distance $=\mathbf{3 2 5}-\mathbf{3 0 0}$

$$
=\mathbf{2 5} \text { miles }
$$

The distance between the cbuses is $\mathbf{2 5}$ miles.
3. A passenger and a freight train leave a city at the same time from the same point and travel in opposite directions on a straight road. The freight train moves at 40 mph slower than the passenger train. What is the speed of the freight train if the trains are $\mathbf{2 6 0}$ miles apart after $\mathbf{3}$ hours $\mathbf{1 5}$ minutes?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the speed of the passenger train. $\mathbf{1 5}$ minutes is a quarter of an hour. The trains travel for 3.25 hours.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Passenger train | $\boldsymbol{x}$ | $\cdot$ | $\mathbf{3 . 2 5}$ | $=$ | $\mathbf{3 . 2 5 x}$ |
| Freight train | $\boldsymbol{x}-\mathbf{4 0}$ | $\cdot$ | $\mathbf{3 . 2 5}$ | $=$ | $\mathbf{3 . 2 5 ( x - 4 0 )}$ |

$\underbrace{\mathbf{3 . 2 . 5}(x-40) \mathrm{mi} . \text { (freight train) }} \quad \underbrace{3.25 x \mathrm{mi} .(\text { passenger train) }}$
Equation:

$$
\begin{aligned}
\text { Distance between trains } & =\text { distance of passenger train } \\
& + \text { distance of freight train } \\
3.25 x+3.25(x-40) & =260 \\
3.25 x+3.25 x-130= & 260 \\
6.5 x-130= & 260 \\
6.5 x-130+130= & 260+130 \\
6.5 x= & 390 \\
\frac{6.5}{6.5}= & \frac{390}{6.5} \\
x & =60
\end{aligned}
$$

The freight train travels at $\mathbf{6 0 - 4 0}=\mathbf{2 0} \mathrm{mph}$.
4. Abe rode a Moped at 35 mph . He left at 10 : 00 A.M. At noon Bill discovered Abe's driver's license and wallet at home. Bill rushed into his Bronco and drove the same route as Abe at $\mathbf{6 5} \mathbf{m p h}$. At what time did Bill catch up with Abe?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the time in hours it took Bill to catch Abe. Abe had traveled $\boldsymbol{x}+\mathbf{2}$ hours.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :--- | :---: | :---: | :---: |
| Abe | $\mathbf{3 5}$ | $\cdot$ | $\boldsymbol{x}+\mathbf{2}$ | $=$ | $\mathbf{3 5}(\boldsymbol{x}+\mathbf{2})$ |
| Bill | $\mathbf{6 5}$ | $\cdot$ | $\boldsymbol{x}$ | $=$ | $\mathbf{6 5 x}$ |


$\overbrace{65 \boldsymbol{x} \text { miles (Bill) }}$
Equation:
Distance traveled by Bill $=$ distance traveled by Abe

$$
\begin{aligned}
65 x & =35(x+2) \\
65 x & =35 x+70 \\
65 x-35 x & =35 x-35 x+70 \\
30 x & =70 \\
\frac{30}{30} x & =\frac{70}{30} \\
x & =2 \frac{1}{3}
\end{aligned}
$$

Bill took 2 hours 20 minutes to catch Abe at 12:00+2:20=14:20 or 2:20 P.M.
5. A plane flies from LAX to ORD at $\mathbf{5 0 0} \mathrm{mph}$ and returns at $\mathbf{3 5 0} \mathrm{mph}$. Find the distance between the two airports if the total travel time is $\mathbf{8 . 5}$ hours.
Preamble:
Let $\boldsymbol{x}$ be the distance between LAX and ORD
Remember that $\boldsymbol{T}=\frac{\boldsymbol{D}}{\boldsymbol{R}}$.

|  | Rate | $\cdot$ | Time | $=$ | distance |
| :---: | :---: | :---: | :---: | :---: | :---: |
| LAX to ORD | $\mathbf{5 0 0}$ | $\cdot$ | $\frac{\boldsymbol{x}}{\mathbf{5 0 0}}$ | $=$ | $\boldsymbol{x}$ |
| ORD to LAX | $\mathbf{3 5 0}$ | $\cdot$ | $\frac{\boldsymbol{x}}{\mathbf{3 5 0}}$ | $=$ | $\boldsymbol{x}$ |

Equation:
Time from LAX to ORD

+ time from ORD to LAX $=$ total time

$$
\begin{aligned}
\frac{x}{500}+\frac{x}{350} & =8.5 \\
\frac{x}{500} \cdot \frac{7}{7}+\frac{x}{350} \cdot \frac{10}{10} & =8.5 \\
\frac{7 x+10 x}{500 \cdot 7} & =8.5 \\
\frac{17 x}{500 \cdot 7} & =8.5 \\
\frac{17 x \cdot 500 \cdot 7}{500 \cdot 7} & =8.5 \cdot 500 \cdot 7 \\
17 x & =8.5 \cdot 500 \cdot 7 \\
x & =\frac{8.5 \cdot 500 \cdot 7}{17} \\
x & =\frac{4,250 \cdot 7}{17} \\
x & =1,750
\end{aligned}
$$

The distance between the two airports is $\mathbf{1 , 7 5 0}$ miles.
6. The speed of light is $\mathbf{1 8 6}, \mathbf{2 8 2} \frac{\text { miles }}{\text { second }}$ and the speed of sound $\mathbf{1 , 1 2 7} \frac{\mathrm{ft}}{\text { second }}$. Santa Monica is about 20 miles from Long Beach.
There are $\mathbf{5 , 2 8 0} \mathrm{ft}$ in a mile.
A bolt of lightning strikes somewhere in Santa Monica. Assume the bolt is seen and the thunder is heard in Long Beach.

How long after the lightning is seen will the thunder be heard in Long Beach?

## Solution:

Preamble:
Sound travels at
$\frac{\mathbf{1 , 1 2 7} \mathrm{ft}}{\text { second }}=\frac{\mathbf{1 , 1 2 7} \mathrm{ft}}{\text { second }} \cdot \frac{\text { mile }}{\mathbf{5 , 2 8 0} \mathrm{ft}}=\frac{\mathbf{0 . 2 1 3 4} \text { miles }}{\text { second }}$

|  | Time | $=$ | Distance | $\div$ | Rate |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Lightning | $\boldsymbol{T}_{\boldsymbol{L}}$ | $=\frac{\mathbf{2 0} \text { miles }}{\mathbf{1}}$ | $\div$ | $\frac{\mathbf{1 8 6 , 2 8 2} \text { miles }}{\text { second }}$ |  |
| Sound | $\boldsymbol{T}_{\boldsymbol{S}}$ | $=$ | $\frac{\mathbf{2 0}}{\mathbf{1}}$ | $\div$ | $\frac{\mathbf{0 . 2 1 3 4} \text { miles }}{\text { second }}$ |

Equation:
$T_{L}=\frac{20 \mathrm{miles}}{1} \cdot \frac{\text { second }}{186,282 \mathrm{miles}}=\mathbf{1 . 0 7 3 6 4} \times 10^{-4} \approx 0.0001$ seconds.
$\boldsymbol{T}_{S}=\frac{\mathbf{2 0} \text { miles }}{\mathbf{1}} \cdot \frac{\text { second }}{\mathbf{0 . 2 1 3 4} \mathrm{miles}}=\mathbf{9 4}$ seconds.
If it could be heard, thunder would be delayed by $\mathbf{9 4}$ seconds $=\frac{\mathbf{9 4} \text { seconds }}{\mathbf{1}} \cdot \frac{\mathbf{1} \text { minute }}{\mathbf{6 0} \text { seconds }}=\mathbf{1}$ minute 34 seconds.

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## Chapter 18

# Translating Oral <br> Expressions/Equations into Math 

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### 18.1 YouTube

https://www.youtube.com/playlist?list=PL0B7894C6168AB527\&feature=view_all

### 18.2 Basics

Some of the problems here are simple. The solution can be worked out fast by quick reasoning. The benefit of these problems is not to find the solution by reasoning, but by learning algebraic steps applicable to more challenging situations.

The general method is not to read a problem first and understand it. Many of my colleagues will disagree with me. I propose creating a preamble in which you write a symbol or set of symbols for each element in the problem.

Look at the end of the problem which asks the question. Let $\boldsymbol{x}$ be the number we are looking for. Write this as the first step in the preamble. Start reading the problem and develop the preamble step by step. Write the symbol(s) for each step on a new line. Finish converting all the steps in the problem into symbols. Now look at your preamble and understand it.

It will be easier to obtain an equation using the symbols from the preamble. Then solve the equation by the method introduced this far for linear equations. (A linear equation has a variable to the first degree (exponent).)

When we learn a new language, we translate starting with a familiar language. Mathematics is a new language and English is a familiar language. Vocabulary is basic.

Words indicating an arithmetic operation:

| Addition | Subtraction | Multiplication |  |
| :--- | :--- | :--- | :--- |
| add | subtract | multiply | Division <br> divide |
| plus | minus | times | quotient |
| more than | less than | of | ratio |
| sum | less | product | fraction |
| increase | difference | twice | into |
| total | decrease | double | per |
| altogether | decrement | triple | half |
| in all | subtracted from | square | third |
| increment | deduct |  |  |
| double | discount |  |  |
|  | take away |  |  |

### 18.3 Examples

Develop each part of the sentence from the smallest detail to the more complicated one.

Example 1:
Add a number to the quotient of $\mathbf{6}$ and the difference of the number and $\mathbf{6}$.

## Solution:

(1a) Add number to quotient of $\mathbf{6}$ and diff. of number and 6.
(1b) Add $\boldsymbol{x}$ to the quotient of $\mathbf{6}$ and the diff. of $\boldsymbol{x}$ and 6.
(1c) Add $\boldsymbol{x}$ to the quotient of $\mathbf{6}$ and $\boldsymbol{x}-\boldsymbol{6}$.
(1d) Add $\boldsymbol{x}$ to $\frac{\mathbf{6}}{\boldsymbol{x}-\mathbf{6}}$.
(1e) $\frac{\mathbf{6}}{\boldsymbol{x}-\mathbf{6}}+\boldsymbol{x}$.

Example 2:
The sum of twice a number and 7 .

## Solution:

(2a) The sum of twice a number and 7 .
(2b) The sum of twice $\boldsymbol{x}$ and $\boldsymbol{7}$.
(2c) The sum of $\mathbf{2 x}$ and 7 .
(2d) $2 \boldsymbol{x}+7$. Note: no parentheses.

Example 3:
Twice the sum of a number and 7 .

## Solution:

(3a) Twice the sum of a number and 7 .
(3b) Twice the sum of $\boldsymbol{x}$ and 7.
(3c) Twice $\boldsymbol{x}+\boldsymbol{7}$.
(3d) $\mathbf{2}(x+7)$.
Note: parentheses are crucial.

Example 4:
The difference of 12 and 4 subtracted from 12 minus 5.

## Solution:

(4a) The difference of 12 and 4 subtracted from 12 minus 5.
(4b) The difference of 12 and 4 subtracted from 7.
(4c) 8 subtracted from 7 .
(4d) $7-8=-1$

Example 5:
The product of $\mathbf{3}$ and triple a number, deducted from the total of one-third the number and $\mathbf{9}$.

## Solution:

(5a) Prod. of $\mathbf{3} \&$ triple num, ded. from total of $\frac{\mathbf{1}}{\mathbf{3}}$ num \& $\mathbf{9}$.
(5b) Prod. of $\mathbf{3} \&$ triple $\boldsymbol{x}$, ded. from total of $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}$ and $\mathbf{9}$.
(5c) Product of $\mathbf{3} \& \mathbf{3} \boldsymbol{x}$, deducted from total of $\frac{\boldsymbol{x}}{\mathbf{3}}$ and $\mathbf{9}$.
(5d) $\mathbf{3} \cdot \mathbf{3} \boldsymbol{x}$, deducted from $\frac{\boldsymbol{x}}{\mathbf{3}}+\mathbf{9}$.
(5e) $\left(\frac{\boldsymbol{x}}{\mathbf{3}}+\mathbf{9}\right)-\mathbf{9} \boldsymbol{x}$.
Example 6:
Mike's score $\boldsymbol{M}$ is $\mathbf{7}$ points more than Peter's score $\boldsymbol{P}$, and Mike's score $\boldsymbol{M}$ is $\mathbf{5}$ points less than Norman's score $\boldsymbol{N}$. What is the difference between Norman's score and Peter's score?

## Solution:

(6a) Mike's score $\boldsymbol{M}$ is $\mathbf{7}$ points more than Peter's score $\boldsymbol{P}$, and Mike's score $\boldsymbol{M}$ is $\boldsymbol{N} \mathbf{- 5}$.
(6b) $\boldsymbol{M}=\boldsymbol{P}+\mathbf{7}, \boldsymbol{M}=\boldsymbol{N} \boldsymbol{- 5}$.
(6c) $M=M$ so $\boldsymbol{P}+\mathbf{7}=\boldsymbol{N}-\mathbf{5}$ or $\boldsymbol{N}-\boldsymbol{P}=\mathbf{1 2}$.

Example 7:
Half your income $\boldsymbol{M}$ is increased by $\mathbf{2 5}$.

## Solution:

(7a) Half your income $\boldsymbol{M}$ is increased by 25.
(7b) $\frac{M}{2}$ is increased by 25.
(7c) $\frac{M}{2}+\mathbf{2 5}$.
Example 8:
The square of a number less $\mathbf{3}$ is doubled.

## Solution:

(8a) The square of a number less $\mathbf{3}$ is doubled.
(8b) The square of a $\boldsymbol{x}-\mathbf{3}$ is doubled.
(8c) $(\boldsymbol{x}-3)^{2}$ is doubled.
(8d) $2(\boldsymbol{x}-\mathbf{3})^{\mathbf{2}}$.
Example 9:
Amount $\boldsymbol{A}$ is number $\boldsymbol{P}$ per $\mathbf{1 0 0}$ of number $\boldsymbol{B}$.

## Solution:

(9a) Amount $\boldsymbol{A}$ is number $\boldsymbol{P}$ per 100 of number $\boldsymbol{B}$.
(9b) Amount $\boldsymbol{A}$ is $\frac{\boldsymbol{P}}{\mathbf{1 0 0}}$ of number $\boldsymbol{B}$.
(9c) Amount $\boldsymbol{A}$ is $\frac{\boldsymbol{P}}{\mathbf{1 0 0}} \cdot \boldsymbol{B}$.
(9d) $\boldsymbol{A}=\frac{\boldsymbol{P}}{\mathbf{1 0 0}} \cdot \boldsymbol{B}$.
Example 10:
The area of a trapezoid $\boldsymbol{A}$ is the fraction with numerator the height $\boldsymbol{h}$, multiplied by the sum of number $\boldsymbol{B}$ and number $\boldsymbol{b}$, and denominator $\mathbf{2}$.

## Solution:

(10a) $\boldsymbol{A}$ is the fraction with numerator $\boldsymbol{h}$, multiplied by number $\boldsymbol{B}$ plus number $\boldsymbol{b}$, and denominator $\mathbf{2}$.
(10b) $\boldsymbol{A}$ is the fraction with numerator $\boldsymbol{h}$, multiplied by $\boldsymbol{B}+\boldsymbol{b})$, and denominator 2 .
(10c) $\boldsymbol{A}=$ fraction with num. $\boldsymbol{h}(\boldsymbol{B}+\boldsymbol{b})$, and den. 2.
(10d) bf The area of a trapezoid $\boldsymbol{A}$ is $\frac{\boldsymbol{h}(\boldsymbol{B}+\boldsymbol{b})}{2}$.
(10e) $\boldsymbol{A}=\frac{\boldsymbol{h}(\boldsymbol{B}+\boldsymbol{b})}{2}$.
Isn't the statement of this example an awful sentence? It is so easy to be misled. Do you see why you need to learn this new universal language called mathematics?

### 18.4 Exercise 18

Translate the verbal statements below into mathematical symbols.

1. Increment the quotient of $\mathbf{6}$ and a number by $\mathbf{9}$. What is the product of the previous result and $\mathbf{7}$ ?
2. The sum of twice a number and $\mathbf{1 1}$.
3. Twice the sum of a number and $\mathbf{1 1 .}$
4. The ratio of a number and 4 , subtracted from 12 minus 5 .
5. The product of $\mathbf{8}$ and four times a number, decreased by the total of one-third the number and $\mathbf{1 9}$.
6. Mike $\boldsymbol{M}$, Norman $\boldsymbol{N}$ and Peter $\boldsymbol{P}$ have $\boldsymbol{x}$ points altogether. $\boldsymbol{M}$ 's score is $\mathbf{2}$ points fewer than their sum. What is Mike's score in terms of the sum of the three players?
7. Half your price $\boldsymbol{P}$ is increased by $\mathbf{2 5}$. This result is divided into $\mathbf{5 2 5}$.
8. Take away 1 from the product of the cube of a number and its reciprocal.
9. The regular price $\boldsymbol{R}$ of a coat was discounted by $\boldsymbol{D}$. The coat was sold for $\$ \boldsymbol{R}-\boldsymbol{D}$. The merchant had $\$ \boldsymbol{L}$ left in his bank account after withdrawing the amount of the sales price of the coat. How much money was left in the account?

## STOP!

1. Increment the quotient of $\mathbf{6}$ and a number by $\mathbf{9}$. What is the product of the previous result and $\mathbf{7}$ ?

## Solution:

(1a) Increment the quotient of $\mathbf{6}$ and a number by $\mathbf{9}$.
(1b) Increment the quotient of $\mathbf{6}$ and $\boldsymbol{x}$ by $\mathbf{9}$.
(1c) Increment $\frac{\mathbf{6}}{\boldsymbol{x}}$ by 9 .
(1d) $\frac{6}{x}+9$. What is the product of the previous result and 7 ?
(1e) $\mathbf{7}\left(\frac{\mathbf{6}}{\boldsymbol{x}}+\mathbf{9}\right)$
Note: The parentheses are crucial
2. The sum of twice a number and $\mathbf{1 1}$.

## Solution:

(2a) The sum of twice a number and 11.
(2b) The sum of twice $\boldsymbol{x}$ and 11.
(2c) The sum of $\mathbf{2 x}$ and 11.
(2d) $\mathbf{2 x}+11$.
Note: no parentheses.
3. Twice the sum of a number and $\mathbf{1 1}$.

## Solution:

(3a) Twice the sum of a number and 11.
(3b) Twice the sum of $\boldsymbol{x}$ and 11.
(3c) Twice $\boldsymbol{x}+\mathbf{1 1}$.
(3d) $2(x+7)$.
Note: parentheses are crucial.
4. The ratio of a number and 4, subtracted from 12 minus 5 .

## Solution:

(4a) Ratio of number and 4, subtracted from 12 minus 5.
(4b) Ratio of $\boldsymbol{x}$ and 4, subtracted from 12 minus 5.
(4c) $\frac{x}{4}$, subtracted from 12 minus 5 .
(4d) $\frac{x}{4}$, subtracted from $12-5=7$.
(4e) $\boldsymbol{7}-\frac{\boldsymbol{x}}{\boldsymbol{4}}$
5. The product of $\mathbf{8}$ and four times a number, decreased by the total of one-third the number and $\mathbf{1 9}$.

Solution:
(5a) Product of $\mathbf{8} \& \mathbf{4}$ times num., decreased by total of one-third num. \& 19.
(5b) Prod. of $\mathbf{8} \& 4$ times $\boldsymbol{x}$, decr. by total of $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x} \& 19$.
(5c) Product of $8 \& 4 \boldsymbol{x}$, decr. by total of $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}$ and 19 .
(5d) $\mathbf{8 ( 4 x})=\mathbf{3 2 x}$, decreased by total of $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}$ and $\mathbf{1 9}$.
(5f) $\mathbf{3 2 x}$, decreased by the total of $\frac{\boldsymbol{x}}{\mathbf{3}}$ and 19 .
(5g) $32 \boldsymbol{x}-\left(\frac{\boldsymbol{x}}{\mathbf{3}}+19\right)$.
Note the last parentheses.
6. Mike $\boldsymbol{M}$, Norman $\boldsymbol{N}$ and Peter $\boldsymbol{P}$ have $\boldsymbol{x}$ points altogether. $\boldsymbol{M}$ 's score is $\mathbf{2}$ points fewer than their sum. What is Mike's score in terms of the sum of the three players?

## Solution:

(6a) Mike $\boldsymbol{M}$, Norman $\boldsymbol{N}$ and Peter $\boldsymbol{P}$ have $\boldsymbol{x}$ points altogether. $\boldsymbol{M}$ 's score is $\mathbf{2}$ points fewer than their sum.
(6b) $\boldsymbol{M}+\boldsymbol{N}+\boldsymbol{P}=\boldsymbol{x} . \boldsymbol{M}$ 's score is $\mathbf{2}$ points fewer than their sum.
(6c) $M$ 's score is 2 points fewer than their sum.
(6d) $\boldsymbol{M}=\boldsymbol{x}-\mathbf{2}$.
7. Half your price $\boldsymbol{P}$ is increased by $\mathbf{2 5}$. This result is divided into $\mathbf{5 2 5}$.

## Solution:

(7a) Half your price $\boldsymbol{P}$ is increased by $\mathbf{2 5}$. This result is divided into $\mathbf{5 2 5}$.
(7b) $\frac{\boldsymbol{P}}{\mathbf{2}}$ is increased by $\mathbf{2 5}$. This result is divided into $\mathbf{5 2 5}$.
(7c) $\frac{\boldsymbol{P}}{2}+25$. This result is divided into 525 .
(7d) $\frac{525}{\frac{P}{2}+25}$.
8. Take away 1 from product of cube of number and its reciprocal.

## Solution:

(8a) Take away 1 from product of cube of num. and its reciprocal.
(8b) Take away $\mathbf{1}$ from the product of cube of $\boldsymbol{x}$ and $\frac{\mathbf{1}}{\boldsymbol{x}}$.
(8c) Take away $\mathbf{1}$ from the product of $\boldsymbol{x}^{\mathbf{3}}$ and $\frac{\mathbf{1}}{\boldsymbol{x}}$.
(8d) Take away 1 from $\frac{x^{3}}{x}=x^{2}$.
(8e) $\boldsymbol{x}^{2}-1$.
9. The regular price $\boldsymbol{R}$ of a coat was discounted by $\boldsymbol{D}$. The coat was sold for $\$ \boldsymbol{R}-\boldsymbol{D}$. The merchant had $\$ \boldsymbol{L}$ left in his bank account after withdrawing the amount of the sales price of the coat. How much money was left in the account?

## Solution:

(9a) The regular price $\boldsymbol{R}$ of a coat was discounted by $\$ \boldsymbol{D}$.
(9b) The coat was sold for $\boldsymbol{\$} \boldsymbol{R}-\boldsymbol{D}$. The merchant had $\$ \boldsymbol{L}$ left in his bank account. He withdrew the same amount as sales price of the coat. How much money was left in the account?
(9b) $\boldsymbol{L}-(\boldsymbol{R}-\boldsymbol{D})=\boldsymbol{L}-\boldsymbol{R}+\boldsymbol{D}$ was left in the account.

## Chapter 19

## Solving Formulas

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### 19.1 YouTube

https://www.youtube.com/playlist?list=PL007CEE47E5DAF8C5\&feature=view_all

### 19.2 General Approach

A literal equation is an equation with letters. The letters stand for numbers. Think of replacing the letters with some numbers of your choice (except for the letter, that is unknown, that you solve for).

Why are literal equations (formulas) useful?
Imagine the following:
You want to solve:
(a)

$$
\begin{aligned}
\frac{x-3}{2}+8 & =1 \\
\frac{x-3}{2}+8-8 & =1-8 \\
\frac{x-3}{2} & =-7 \\
x-3 & =-7(2) \\
x-3 & =-14 \\
x-3+3 & =-14+3 \\
x & =-11
\end{aligned}
$$

Repeat the same steps with different numbers:
(b) $\quad \frac{x-4}{3}+1=7$

$$
\begin{aligned}
\frac{x-4}{3}+1-1 & =7-1 \\
\frac{x-4}{3} & =6 \\
x-4 & =6(3) \\
x-4 & =18 \\
x-4+4 & =18+4 \\
x & =22
\end{aligned}
$$

And again:
(c)

$$
\begin{aligned}
\frac{x-0.3}{0.2}+0.8 & =0.4 \\
\frac{x-0.3}{0.2}+0.8-0.8 & =0.4-0.8 \\
\frac{x-0.3}{0.2} & =-0.4 \\
x-0.3 & =-0.4(0.2) \\
x-0.3 & =-0.08 \\
x-0.3+0.3 & =-.08+0.3 \\
x & =0.22
\end{aligned}
$$

Every time we are performing the same steps.
Now let's develop a general formula identical to the three examples but has letters instead of numbers.

$$
\begin{aligned}
\frac{(x-a}{b}+c & =d \\
\frac{x-a}{b}+c-c & =d-c \\
\frac{(x-a}{b} & =d-c \\
(x-a & =b(d-c) \\
(x-a+a & =b(d-c)+a \\
(x & =b(d-c)+a
\end{aligned}
$$

Evaluate $\boldsymbol{x}=\boldsymbol{b}(\boldsymbol{d}-\boldsymbol{c})+\boldsymbol{a}$ for $a=3, b=2, c=8$ and $\boldsymbol{d}=1$.
$x=2(1-8)+3=2(-7)+3=-14+3=-11$
Evaluate $x=b(d-c)+a$ for $a=4, b=3, c=1, d=7$.
$x=3(7-1)+4=3(6)+4=18+4=22$
Evaluate $\boldsymbol{x}=\boldsymbol{b}(\boldsymbol{d}-\boldsymbol{c})+\boldsymbol{a}$ for $\boldsymbol{a}=0.3, b=0.2, c=0.8, d=0.4$.
$x=0.2(0.4-0.8)+0.3=0.2(-0.4)+0.3=-0.08+0.3=0.22$
The three results above correspond to the results obtained in (a), (b) and (c). Evaluating an expression is usually easier than developing a formula.

Did you notice that the steps in developing the formula
$\boldsymbol{x}=\boldsymbol{b}(\boldsymbol{d}-\boldsymbol{c})+\boldsymbol{a}$ were identical to the steps in (a), (b) and (c)?
When you are faced with developing a solution to a literal equation, write the original equation next to a similar equation with numbers of your choice. Each step taken with the numbers corresponds to the step taken in your literal equation.

### 19.3 Examples

Example 1:
Solve $\boldsymbol{P}=\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}$ for $\boldsymbol{a}$.

## Solution:

$$
\begin{array}{rlrlrl}
P & =a+b+c & \mathbf{a} & =a+4+5 & & \text { Original equation } \\
P & =a+(b+c) & \mathbf{a}+\boldsymbol{a}+(9) & & \text { add } 4 \text { and } 5 . \\
P-(b+c) & =a & 7-(9) & =a & & \text { add } \mathbf{4} \text { and } \mathbf{5} .
\end{array}
$$

Thus $\boldsymbol{a}=\boldsymbol{P}-(\boldsymbol{b}+\boldsymbol{c})$
Example 2:
Solve $\boldsymbol{A}=\boldsymbol{L} \boldsymbol{W}$ for $\boldsymbol{L}$.

## Solution:

$$
\begin{aligned}
A & =\Delta L W \\
\frac{A}{W} & =\frac{\boldsymbol{L}}{\boldsymbol{L}} \boldsymbol{W} \\
\frac{A}{W} & =L \cdot L \\
\frac{7}{4}=\frac{L \cdot 4}{4} & \text { Divide by } 4, \text { divide by } W
\end{aligned}
$$

Thus $\boldsymbol{L}=\frac{\boldsymbol{A}}{\boldsymbol{W}}$
Example 3:
Solve $\boldsymbol{P}=\mathbf{2 L}+\mathbf{2 W}$ for $\boldsymbol{L}$.

## Solution:

$$
\begin{aligned}
& P=2 L+2 W \quad 7=2 L+2(7) \text { Original equation } \\
& P-2 W=2 L \quad 7-2(7)=2 L \quad \text { Subtract } 2(7), \text { subtract } 2 W \\
& \frac{P-2 W}{2}=\frac{2 L}{2} \quad \frac{7-2(7)}{2}=\frac{2 L}{2} \quad \text { Divide by } 2 \text {, divide by } 2 \\
& \frac{P-2 W}{2}=L \quad \frac{7-2(7)}{2}=L \quad \text { Divide by } 2 \text {, divide by } 2
\end{aligned}
$$

Thus $L=\frac{P-2 W}{2}=\frac{P}{2}-W$

Example 4:
Solve $\boldsymbol{A}=\frac{\boldsymbol{H}}{\mathbf{2}}\left(\boldsymbol{b}_{1}+\boldsymbol{b}_{2}\right)$ for $\boldsymbol{b}_{\mathbf{1}}$.

## Solution:

$$
\begin{aligned}
& A=\frac{H}{2}\left(\sqrt[b_{1}]{ }+b_{2}\right) \quad 7=\frac{9}{2}\left(b_{1}+3\right) \quad \text { Original equation } \\
& 2 A=\frac{2 H}{2}\left(\boxed{b_{1}}+b_{2}\right) \\
& 2 A=H\left(\boxed{b_{1}}+b_{2}\right) \\
& \frac{2 A}{H}=\frac{H\left(\boxed{b_{1}}+b_{2}\right)}{H} \\
& \frac{2 A}{H}=b_{1}+b_{2} \\
& 2(7)=\frac{2 \cdot 9}{2}\left(b_{1}+3\right) \quad \text { Multiply by } 2 \text {, multiply by } 2 \\
& \mathbf{2 ( 7 )}=\mathbf{9}\left(b_{1}+\mathbf{3}\right) \quad \text { Reduce, reduce } \\
& \frac{2(7)}{9}=\frac{9\left(b_{1}+3\right)}{9} \quad \text { Divide by } \mathbf{9}, \text { divide by } \boldsymbol{H} \\
& \frac{2(7)}{9}=b_{1}+3 \quad \text { Divide by } 9, \text { divide by } \boldsymbol{H} \\
& \frac{2 A}{H}-b_{2}=b_{1} \quad \frac{2(7)}{9}-3=b_{1} \quad \text { Subtract } 3, \text { subtract } b_{2} \\
& \text { Thus } \boldsymbol{b}_{1}=\frac{\mathbf{2 A}}{\boldsymbol{H}}-\boldsymbol{b}_{\mathbf{2}}
\end{aligned}
$$

Example 5:
Solve $a^{2}+b^{2}=c^{2}$ for $a$

## Solution:

$$
\begin{array}{rl||rll}
a^{2}+b^{2} & =c^{2} & & \text { Original equation } \\
a^{2}+5^{2} & =3^{2} & c^{2}-b^{2} \\
a^{2} & =3^{2}-5^{2} & \text { Subtract } 5^{2}, \text { subtract } b^{2} \\
a & =\sqrt{c^{2}-b^{2}} & a & =\sqrt{3^{2}-5^{2}} & \text { Sq. root, sq. root, subtract } b^{2}
\end{array}
$$

Thus $a=\sqrt{c^{2}-b^{2}}$

Example 6:
Solve $\boldsymbol{S}=\boldsymbol{a} \boldsymbol{R H}+\boldsymbol{b} \boldsymbol{R}$ for $\boldsymbol{R}$

## Solution:

$$
\begin{aligned}
S & \left.\left.=a \boxed{R} H+b \boxed{R}| | \begin{array}{rl}
9 & =7 R(5)+3 R \text { Original equation } \\
S & =R(a H+b) \\
\frac{S}{a H+b} & =\frac{R[7(5)+3] \text { Factor } R, \text { factor } R}{a H+b} \\
\frac{S}{a H+b} & =R
\end{array} \right\rvert\, \begin{array}{rl}
\frac{S}{7(5)+3} & =\frac{R[7(5)+3]}{7(5)+3} \div \text { by } 7(5)+3 \div \text { by } a H+b \\
\frac{9}{7(5)+3} & =R
\end{array}\right) \quad \text { by } 7(5)+3 \div \text { by } a H+b
\end{aligned}
$$

Thus $\boldsymbol{R}=\frac{\boldsymbol{S}}{\boldsymbol{a} \boldsymbol{H}+\boldsymbol{b}}$
Example 7:
Solve $\boldsymbol{m}=\frac{\boldsymbol{y}_{2}-\boldsymbol{y}_{1}}{\boldsymbol{x}_{2}-\boldsymbol{x}_{1}}$ for $\boldsymbol{x}_{\mathbf{2}}$


Example 8:
Solve $\boldsymbol{y}=\boldsymbol{m} \boldsymbol{x}+\boldsymbol{b}$ for $\boldsymbol{x}$.

## Solution:

Thus $\boldsymbol{x}=\frac{\boldsymbol{y}-\boldsymbol{b}}{\boldsymbol{m}}$
Example 9:
Solve $\frac{1}{a} \cdot t+\frac{1}{b} \cdot t=1$ for $t$.

## Solution:

$$
\begin{aligned}
& \frac{1}{a} \cdot t+\frac{1}{b} \cdot t=1 \quad \frac{1}{9} \cdot t+\frac{1}{7} \cdot t=1 \\
& \text { Original equation } \\
& \frac{7(9)}{9} \cdot t+\frac{7(9)}{7} \cdot t=7(9) \\
& \text { Multiply by } 7(9) \text {, multiply by } a b \\
& (7+9) t=7(9) \\
& \text { Factor } \boldsymbol{t} \text {, factor } \boldsymbol{t} \\
& \frac{(7+9)}{7+9} t=\frac{7(9)}{7+9} \\
& \text { Divide by } \mathbf{7}+\mathbf{9} \text {, divide by } \boldsymbol{a}+\boldsymbol{b} \\
& t=\frac{7(9)}{7+9}
\end{aligned}
$$

Thus $t=\frac{a b}{a+b}$

### 19.4 Exercise 19

1. Solve $\boldsymbol{D}=\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{p}$ for $\boldsymbol{m}$.
2. Solve $\boldsymbol{Q}=\boldsymbol{x} \boldsymbol{y}$ for $\boldsymbol{x}$.
3. Solve $\boldsymbol{F}=\mathbf{2 a}+\mathbf{2} \boldsymbol{b}$ for $\boldsymbol{a}$.
4. Solve $\boldsymbol{Q}=\frac{\boldsymbol{w}}{\mathbf{2}}(\boldsymbol{c}+\boldsymbol{d})$ for $\boldsymbol{c}$.
5. Solve $\boldsymbol{x}^{2}+y^{2}=z^{2}$ for $\boldsymbol{x}$
6. Solve $\boldsymbol{T}=\boldsymbol{a} \boldsymbol{P} \boldsymbol{L}+\boldsymbol{b} \boldsymbol{P}$ for $\boldsymbol{P}$
7. Solve $\boldsymbol{m}=\frac{\boldsymbol{a}-\boldsymbol{b}}{\boldsymbol{c}-\boldsymbol{d}}$ for $\boldsymbol{c}$
8. Solve $\boldsymbol{s}=\boldsymbol{v} \boldsymbol{h}+\boldsymbol{a}$ for $\boldsymbol{h}$.
9. Solve $\frac{1}{\boldsymbol{m}} \cdot \boldsymbol{x}+\frac{\mathbf{1}}{\boldsymbol{n}} \cdot \boldsymbol{x}=\mathbf{1}$ for $\boldsymbol{x}$.

## STOP!

1. Solve $\boldsymbol{D}=\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{p}$ for $\boldsymbol{m}$.

## Solution:



Thus $\boldsymbol{m}=\boldsymbol{D}-(\boldsymbol{n}+\boldsymbol{p})$
2. Solve $\boldsymbol{Q}=\boldsymbol{x} \boldsymbol{y}$ for $\boldsymbol{x}$.

## Solution:

| $Q=\boxed{x} y$ | $7=4 x$ | Original equation |
| :---: | :---: | :---: |
| $\frac{Q}{y}=\frac{\underset{x}{x} y}{y}$ | $\frac{7}{4}=\frac{4 x}{4}$ | Divide by 4 , divide by $\boldsymbol{y}$. |
| $\frac{Q}{y}=\boldsymbol{x}$ | $\frac{7}{4}=x$ | Divide by 4 , divide by $\boldsymbol{y}$. |

Thus $\boldsymbol{x}=\frac{\boldsymbol{Q}}{\boldsymbol{y}}$
3. Solve $\boldsymbol{F}=\mathbf{2 a}+\mathbf{2 b}$ for $\boldsymbol{a}$.

## Solution:

$$
\begin{aligned}
& \begin{array}{rlrl}
F & =2 a+2 b & & =2 a+2(7) \\
& \text { Orig. eq. } \\
F-2 b & =2 a \\
\frac{F-2 b}{2} & =\frac{2 a}{2} \\
\frac{F-2(7)}{}=2 a & \text { Minus } 2(7), \& 2 W \\
\frac{7-2(7)}{2} & =\frac{2 a}{2} & \div \text { by } 2, \div \text { by } 2 \\
\frac{7-2(7)}{2} & =a & \div \text { by } 2, \div \text { by } 2
\end{array} \\
& \text { Thus } a=\frac{F-2 b}{2}=\frac{F}{2}-b
\end{aligned}
$$

4. Solve $\boldsymbol{Q}=\frac{\boldsymbol{w}}{\mathbf{2}}(\boldsymbol{c}+\boldsymbol{d})$ for $\boldsymbol{c}$.

## Solution:

$$
\begin{aligned}
& \text { Thus } c=\frac{2 Q}{w}-d
\end{aligned}
$$

5. Solve $\boldsymbol{x}^{2}+y^{2}=\boldsymbol{z}^{2}$ for $\boldsymbol{x}$

## Solution:

$$
\begin{array}{rlrl}
x^{2}+y^{2} & =z^{2} & \text { Original equation } \\
x^{2} & =z^{2}-y^{2} \\
x & =\sqrt{z^{2}-y^{2}} & 3^{2} & x^{2}=3^{2}-5^{2} \\
x & =\sqrt{3^{2}-5^{2}} \sqrt{ }, \sqrt{ }, \text { Minus } 5^{2}, \text { minus } y^{2} \\
x & &
\end{array}
$$

Thus $\boldsymbol{x}=\sqrt{z^{2}-\boldsymbol{y}^{2}}$
Note that the square root of $\mathbf{3}^{\mathbf{2}}-5^{\mathbf{2}}$ is not a real number. This means that not all real numbers can be used. We'll come back to this topic later when we discuss domains of functions.
6. Solve $\boldsymbol{T}=\boldsymbol{a} \boldsymbol{P} \boldsymbol{L}+\boldsymbol{b} \boldsymbol{P}$ for $\boldsymbol{P}$

Solution:

$$
\begin{aligned}
& T = a \widehat { P } L + b \longdiv { P } \quad 9 = 7 P ( 5 ) + 3 P \\
& \text { Original equation } \\
& 9=P[7(5)+3] \\
& \text { Factor } \boldsymbol{P} \text {, factor } \boldsymbol{P} \\
& \frac{T}{a L+b}=\frac{\boxed{P}(a L+b)}{a L+b} \frac{9}{7(5)+3}=\frac{P[7(5)+3]}{7(5)+3} \\
& \div \text { by } 7(5)+3, \div \text { by } a L+b \\
& \frac{T}{a L+b}=P \\
& \text { Thus } \boldsymbol{P}=\frac{\boldsymbol{T}}{\boldsymbol{a L}+\boldsymbol{b}} \\
& \frac{9}{7(5)+3}=P \\
& \text { in } \div \text { by } 7(5)+3, \div \text { by } \boldsymbol{a} \boldsymbol{L}+\boldsymbol{b}
\end{aligned}
$$

7. Solve $\boldsymbol{m}=\frac{\boldsymbol{a}-\boldsymbol{b}}{\boldsymbol{c}-\boldsymbol{d}}$ for $\boldsymbol{c}$

$$
\begin{aligned}
& \text { Solution: } \\
& \text { Solution: } \\
& m=\frac{a-b}{\square c-d} \\
& m(\boxed{c}-d)=a-b \\
& m \boxed{c}-m d=a-b \\
& m \boxed{c}=a-b+m d \\
& \frac{m \boxed{c}}{m}=\frac{a-b+m d}{m} \\
& c=\frac{a-b+m d}{m} \\
& \text { Thus } \boldsymbol{c}=\frac{\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{m d}}{\boldsymbol{m}} \\
& 9=\frac{7-5}{c-3} \\
& \text { Original equation } \\
& 9(c-3)=7-5 \\
& \text { Times } \mathbf{7 - 5} \text {, times } \boldsymbol{c} \boldsymbol{- d} \\
& 9 c-9(3)=7-5 \\
& \text { Distribute } \boldsymbol{m} \text {, distribute } \mathbf{9} \\
& 9 c=7-5+9(3) \\
& \text { Add } \boldsymbol{m d} \text {, add } \mathbf{9 ( 3 )} \\
& \frac{9 c}{9}=\frac{7-5+9(3)}{9} \\
& \text { Divide by } \boldsymbol{m} \text {, divide by } \mathbf{9} \\
& c=\frac{7-5+9(3)}{9} \\
& \text { Reduce }
\end{aligned}
$$

8. Solve $\boldsymbol{s}=\boldsymbol{v} \boldsymbol{h}+\boldsymbol{a}$ for $\boldsymbol{h}$.

## Solution:

| $s=v h+a$ | $9=7 h+5$ | Original equation |
| :---: | :---: | :---: |
| $s-a=v h$ | $9-5=7 h$ | Subtract 5, subtract $\boldsymbol{a}$ |
| $\frac{s-a}{v}=\frac{v h}{v}$ | $\frac{9-5}{7}=\frac{7 h}{7}$ | Divide by 7 , divide by $\boldsymbol{v}$ |
| $\frac{s-a}{v}=h$ | $\frac{9-5}{7}=h$ | Reduce, reduce |

Thus $h=\frac{s-a}{v}$
9. Solve $\frac{1}{m} \cdot x+\frac{1}{n} \cdot x=1$ for $x$.

## Solution:

$$
\begin{aligned}
& \frac{1}{m} \cdot x+\frac{1}{n} \cdot x=1 \\
& \frac{1}{9} \cdot x+\frac{1}{7} \cdot x=1 \\
& \text { Original equation } \\
& \frac{m n}{m} \cdot x+\frac{m n}{n} \cdot x=m n \\
& n \cdot x+m \cdot x=m n \\
& (n+m) x=m n \\
& \frac{(n+m)}{m+n}-x=\frac{m n}{m+n} \\
& x=\frac{m n}{m+n} \\
& \text { Thus } \boldsymbol{x}=\frac{\boldsymbol{m n}}{\boldsymbol{m}+\boldsymbol{n}}
\end{aligned}
$$

## Chapter 20

## Applications with Percents

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### 20.1 YouTube

https://www.youtube.com/playlist?list=PL7B615ED7668DC9A3\&feature=view_all

### 20.2 Move the Decimal Point

We know we need to move the decimal point two places. But do we move to the right or the left?
Remember that "per" refers to division ("of" refers to multiplication). Percent (think of Per (space) Cent) with "cent" dealing with 100 .

5 percent then is 5 per 100 or
$\frac{5}{100}=\frac{5}{100} \cdot 1=\frac{5}{100} \cdot \frac{0.01}{0.01}=\frac{0.05}{1.00}=0.05$ or five hundredths.
Similarly the decimal $\mathbf{0 . 7 3}$ becomes $\frac{\mathbf{0 . 7 3}}{\mathbf{1}}=\frac{\mathbf{0 . 7 3}}{\mathbf{1 . 0 0}} \cdot \frac{\mathbf{1 0 0}}{\mathbf{1 0 0}}=\frac{\mathbf{7 3}}{\mathbf{1 0 0}}$.
The phrase

# Amount 1S percent(decimal) Of Base 

$$
A=P_{D} \cdot B
$$

Percent can also be used in ratios. $\frac{\boldsymbol{P}}{\mathbf{1 0 0}}=\frac{\boldsymbol{A}}{\boldsymbol{B}}$. There is a danger
that the numbers for $\boldsymbol{A}$ and $\boldsymbol{B}$ get reversed. The possibility of a mistake is lessened by remembering that the denominator $\boldsymbol{B}$ is associated with "of" (percent of base).
Let's illustrate the usefulness of the boxed in phrase.
Case 1:


Case 2:


Case 3:


### 20.3 Examples

Example 1:
You need to take your California written driver's license test. It consists of $\mathbf{3 6}$ questions. You must answer at least $\mathbf{3 0}$ questions correctly in order to pass. What percent of the questions must you answer correctly?

## Solution:

The number of correct answers is what percent of the total number of questions?
$30=P_{D}(36)$ or $P_{D}=\frac{30}{36} \approx 0.0 .83=83 \%$
Example 2:
A pair of sun glasses was discounted $\mathbf{2 0 \%}$ of the regular price. It sold for $\mathbf{\$ 1 6 8}$. What was the regular price (before the discount)? - Round to the nearest dollar.

## Solution:

Let $\boldsymbol{x}$ be the regular price.
The discount is $\mathbf{2 0 \%}$ of the regular price. $\boldsymbol{D}=\mathbf{0 . 2 \boldsymbol { x }}$.
The sales price is $\boldsymbol{x}-\boldsymbol{D}=\boldsymbol{x}-\mathbf{0 . 2 \boldsymbol { x }}=\mathbf{0 . 8 x}$.
$168=0.8 x$ or $x=\frac{168}{0.8}=\$ 210$
The regular price of the sun glasses was $\$ \mathbf{2 1 0}$.
Example 3:
On Tuesday a restaurant served $\mathbf{1 0 \%}$ of $\mathbf{2 0}$ meals with potatoes for breakfast. $\mathbf{4 0 \%}$ of $\mathbf{4 0}$ lunch meals had potatoes. Finally $\mathbf{8 0 \%}$ of the $\mathbf{6 0}$ dinner meals served potatoes. What was the percent of meals with potatoes served on Tuesday?

## Solution:

Preamble:
$\mathbf{0 . 1}(\mathbf{2 0})=\mathbf{2}$ breakfast meals were ordered with potatoes,
$\mathbf{0 . 4}(40)=16$ lunch meals called for potatoes,
and $\mathbf{0 . 8 ( 6 0 )}=\mathbf{5 4}$ dinner meals had potatoes.
A total of $\mathbf{2}+\mathbf{1 6}+\mathbf{5 4}=\mathbf{7 2}$ meals were consumed with potatoes.
The total number of meals was $\mathbf{2 0}+\mathbf{4 0}+\mathbf{6 0}=\mathbf{1 2 0}$ meals.
Equation:
$\mathbf{7 2}$ meals served with potatoes is what percent of $\mathbf{1 2 0}$ meals served?
Let $\boldsymbol{x}$ be the percent (as a decimal) of meals served with potatoes.

$$
\begin{array}{rlr}
72 & =x(120) \\
\frac{72}{120} & =\frac{120}{120} x \\
\frac{6 \cdot 12}{10 \cdot 12} & =x \quad \text { which leads to } x=\frac{6}{10}=\frac{60}{100}=60 \%
\end{array}
$$

$\mathbf{6 0 \%}$ of the meals were served with potatoes.

### 20.4 Exercise 20

1. You need to take your New York written driver's license test. It consists of $\mathbf{2 0}$ question. You must answer at least $\mathbf{1 4}$ questions correctly in order to pass. What percent of the questions must you answer correctly?
2. A mattress was discounted $\mathbf{2 2 \%}$ of the regular price. It sold for $\mathbf{\$ 2 5 7}$. What was the regular price (before the discount)? - Round to the nearest dollar.
3. On Saturday a movie was seen by $\mathbf{4 0 \%}$ children out of $\mathbf{7 0}$ patrons at the morning show. $\mathbf{2 5 \%}$ of the $\mathbf{3 2}$ patrons at the matinee were children. $\mathbf{1 0 \%}$ of the $\mathbf{6 0}$ evening patrons were children. What percent of children saw the movie on Saturday? Round to the nearest percent.

## STOP!

1. You need to take your New York written driver's license test. It consists of $\mathbf{2 0}$ question. You must answer at least 14 questions correctly in order to pass. What percent of the questions must you answer correctly?

## Solution:

The number of correct answers is what percent of the total number of questions?
$14=P_{D}(20)$ or $P_{D}=\frac{14}{20} \approx 0.7=70 \%$
2. A mattress was discounted $\mathbf{2 2 \%}$ of the regular price. It sold for $\mathbf{\$ 2 5 7}$. What was the regular price (before the discount)? - Round to the nearest dollar.

## Solution:

Let $\boldsymbol{x}$ be the regular price.
The discount is $\mathbf{2 2 \%}$ of the regular price. $\boldsymbol{D}=\mathbf{0 . 2 2 x}$.
The sales price is $\boldsymbol{x}-\boldsymbol{D}=\boldsymbol{x}-\mathbf{0 . 2 2 x}=\mathbf{0 . 7 8 x}$.
$257=0.78 x$ or $x=\frac{257}{\mathbf{0 . 7 8}}=\$ 329$
The regular price of the mattress was $\$ \mathbf{3 2 9}$.
3. On Saturday a movie was seen by $\mathbf{4 0 \%}$ children out of $\mathbf{7 0}$ patrons at the morning show. $\mathbf{2 5 \%}$ of the $\mathbf{3 2}$ patrons at the matinee were children. $\mathbf{1 0 \%}$ of the $\mathbf{6 0}$ evening patrons were children. What percent of children saw the movie on Saturday?

## Solution:

Preamble:
$\mathbf{0 . 4 0}(\mathbf{7 0})=28$ children attended the morning show,
$\mathbf{0 . 2 5 ( 3 2 )}=8$ children saw the afternoon show,
and $\mathbf{0 . 1 ( 6 0 )}=\mathbf{6}$ were children at in the evening.

A total of $28+8+6=42$ children attended the movie.
The total number of patrons was $\mathbf{7 0}+\mathbf{3 2 + 6 0}=\mathbf{1 6 2}$. Round to the nearest percent. Equation:
42 children is what percent of $\mathbf{1 6 2}$ patrons?
Let $\boldsymbol{x}$ be the percent (as a decimal) of children.

$$
\begin{aligned}
42 & =x(162) \\
\frac{42}{162} & =\frac{162}{162} x \\
x & =0.259 \approx 26 \%
\end{aligned}
$$

$\mathbf{2 6 \%}$ of the patrons were children.

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## Chapter 21

## Solving Linear Inequalities and Applications

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### 21.1 YouTube

https://www.youtube.com/playlist?list=PLAD46E0C15CC6325E\&feature=view_all

### 21.2 Applications

The procedure for solving linear inequalities is identical to solving linear equations.
All the "rules" that apply to equations also apply to inequalities with the exception of multiplying or dividing by a negative number.

Add the same number to both sides of an equation (inequality).
Subtract the same number from both sides of an equation (inequality).
Multiply both sides of an equation (inequality) by the same positive number.
Multiply both sides of an equation (inequality) by the same negative number and reverse the direction of the inequality sign.

Divide both sides of an equation (inequality) by the same positive number.
Divide both sides of an equation (inequality) by the same negative number and reverse the direction of the inequality sign.

The following words of equality or inequality should also be understood (consider using only integers in the statements before the table below):
$\boldsymbol{x}$ is more than 3: 4, 5, $6 \ldots$
$\boldsymbol{x}$ is less than $\mathbf{3}: \ldots, \mathbf{- 2}, \mathbf{- 1}, \mathbf{0}, \mathbf{1}, \mathbf{2}$
$\boldsymbol{x}$ is not more than $\mathbf{3}: \ldots, \mathbf{- 2}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}$
$\boldsymbol{x}$ is not less than $\mathbf{3}$ : $\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6} \ldots$
$\boldsymbol{x}$ is at least 3: 3, 4, 5, $6 \ldots$
$\boldsymbol{x}$ is at most $\mathbf{3}: \ldots-2,-1, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}$

| Equations |  | Inequalities |
| :---: | :---: | :---: |
| Add 7 <br> to both sides | $\begin{aligned} 5 & =2+3 \\ 5+7 & =2+3+7 \end{aligned}$ | $\begin{aligned} & 9>2+5 \\ & 6+7>2+3+7 \end{aligned}$ |
| Subtract 7 <br> from both sides | $\begin{aligned} 5 & =2+3 \\ 5-7 & =2+3-7 \end{aligned}$ | $\begin{gathered} 6>2+3 \\ 6-7>2+3-7 \end{gathered}$ |
| $\text { Multiply }+7$ <br> on both sides | $\begin{gathered} 5=2+3 \\ 5(+7)=(2+3)(+7) \end{gathered}$ | $\begin{gathered} 6>2+3 \\ 6(+7)>(2+3)(+7) \end{gathered}$ |
| Multiply -7 on both sides | $\begin{gathered} 5=2+3 \\ 5(-7)=(2+3)(-7) \end{gathered}$ | $\begin{gathered} 6>2+3 \\ 6(-7)<(2+3)(-7) \end{gathered}$ |
| Divide +5 <br> on both sides | $\begin{aligned} 5 & =2+3 \\ \frac{5}{+5} & =\frac{2+3}{+5} \end{aligned}$ | $\begin{gathered} 10>2+3 \\ \frac{10}{+5}>\frac{2+3}{+5} \end{gathered}$ |
| Divide - 5 on both sides | $\begin{aligned} 5 & =2+3 \\ \frac{5}{-5} & =\frac{2+3}{-5} \end{aligned}$ | $\begin{gathered} 10>2+3 \\ \frac{10}{-5}<\frac{2+3}{-5} \end{gathered}$ |

### 21.3 Examples

Example 1:
Harry rents a car from Drive-A-Wreck for one day. The daily charge is $\$ 20$ and he pays $\mathbf{8}$ cents per mile. His budget is $\mathbf{\$ 3 3}$. At most how many miles can he drive?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the maximum number of miles Harry can drive. The cost for mileage is
$\frac{\boldsymbol{x} \text { miles }}{\boldsymbol{1}} \cdot \frac{\mathbf{0 . 0 8} \text { dollars }}{\boldsymbol{1} \text { mile }}=\mathbf{0 . 0 8 x}$ dollars.
The total cost is $\$ \mathbf{2 0}+\mathbf{0 . 0 8 x}$
Equation:

$$
\begin{aligned}
\text { cost } & \leq \text { budget } \\
0.08 x+20 & \leq 33 \\
0.08 x+20-20 & \leq 33-20 \\
0.08 x & \leq 13 \\
\frac{0.08}{0.08} x & \leq \frac{13}{0.08} \\
x & \leq \frac{1300}{8} \\
x & \leq 162
\end{aligned}
$$

Harry can drive at most 162 miles.
Example 2:
A professor has three shelves to hold books. The three shelves measure $\mathbf{4 0}, \mathbf{6 0}$, and $\mathbf{7 5} \mathrm{cm}$ respectively. The professor has books at varying thicknesses: $\mathbf{3 0}$ books at $\frac{1}{2} \mathrm{~cm}, 40$ at $\mathbf{1} \mathrm{cm}, 50$ at $\mathbf{1} \frac{1}{2} \mathrm{~cm}$ and 20 at $\mathbf{2} \mathrm{cm}$. At least how many books can be put on a shelf if the shelf must be completely filled?

## Solution:

The professor wants to put as many books as possible on one shelf. Pick the longest shelf ( $\mathbf{7 5} \mathrm{cm}$ ) and the books with the least thicknesses first.

All $\mathbf{3 0}$ books at $\frac{\mathbf{1}}{\mathbf{2}} \mathrm{cm}$ thickness go on that shelf. The books take up $\mathbf{3 0} \cdot \frac{\mathbf{1}}{\mathbf{2}}=\mathbf{1 5} \mathrm{cm}$. That leaves
$\mathbf{7 5}-\mathbf{1 5}=\mathbf{6 0} \mathrm{cm}$ for the next lower thickness of $\mathbf{1} \mathrm{cm}$. All $\mathbf{4 0}$ of them go on the shelf. This takes up $\mathbf{4 0}$ cm and leaves $\mathbf{6 0 - 4 0}=\mathbf{2 0} \mathrm{cm}$.

Let $\boldsymbol{x}$ be the number of books at $\mathbf{1} \frac{\mathbf{1}}{\mathbf{2}}=\frac{\mathbf{3}}{\mathbf{2}} \mathrm{cm}$.
$\frac{\boldsymbol{x} \text { books }}{1} \cdot \frac{3 \mathrm{~cm}}{2 \text { book }}=1.5 \boldsymbol{x} \mathrm{~cm}$.
$1.5 x \leq 20$ or $x \leq \frac{20}{1.5}$ so $\boldsymbol{x}=13$. These 13 books (out of 50 )
take up $13 \cdot \mathbf{1 . 5}=\mathbf{1 9 . 5} \mathrm{cm}$ of shelf space. That leaves room for just one more book of $\mathbf{0 . 5} \mathrm{cm}$ thickness. Unfortunately all 40 books have been placed already. We can replace one book of thickness $\mathbf{1 . 5} \mathrm{cm}$ by one of thickness 2 cm and the shelf will be completely filled.

The number of books on the shelf is $\mathbf{3 0}+\mathbf{4 0}+\mathbf{1 3}=\mathbf{8 3}$.
At least $\mathbf{8 3}$ books have been shelved. At most $\mathbf{8 3}$ books can be placed one shelf.
Example 3:
Solve $2 x+3<15$.

## Solution:

| Inequality | Equation | Interval notation for the inequality | Graph for the inequality |
| :---: | :---: | :---: | :---: |
| $2 x+3<15$ | $2 x+3=15$ |  |  |
| $2 x+3-3<15-3$ | $2 x+3-3=15-3$ |  |  |
| $2 x<12$ | $2 x=12$ | $\boldsymbol{x}$ is less than 6 |  |
| $\underline{2 x}<\underline{12}$ | $\underline{2 x}=\underline{12}$ |  | $)$ |
| $\overline{2}<\frac{1}{2}$ | $2-\frac{1}{2}$ |  | - 6 |
| $x<6$ | $x=6$ | $(-\infty, 6)$ | $\bigcirc 6$ |

Example 4:
Solve $2 x+3 \leq 15$.

## Solution:

| Inequality $2 x+3 \leq 15$ | Equation $2 x+3=15$ | Interval notation the inequality | Graph for the inequality |
| :---: | :---: | :---: | :---: |
| $2 x+3-3 \leq 15-3$ | $2 x+3-3=15-3$ | $\boldsymbol{x}$ not more than $\mathbf{6}$ |  |
| $2 x \leq 12$ | $2 x=12$ | $\boldsymbol{x}$ is $\leq$ to 6 |  |
| $\frac{2 x}{2} \leq \frac{12}{2}$ | $\frac{2 x}{2}=\frac{12}{2}$ | $\boldsymbol{x}$ is at most 6 | $\xrightarrow{+} 6 \longrightarrow x$ |
| $x \leq 6$ | $x=6$ | $(-\infty, 6]$ | $\underline{\longrightarrow} x$ |

Example 5:
Solve $2 x+3<3 x+15$.

## Solution:

| Inequality | Equation | Int. notation | Graph for the inequality |
| :---: | :---: | :---: | :---: |
| $2 x+3<3 x+15$ | $2 x+3=3 x+15$ |  |  |
| $2 x+3-3<3 x+15-3$ | $2 x+3-3=3 x+15-3$ |  |  |
| $2 x<3 x+12$ | $2 x=3 x+12$ |  |  |
| $2 x-3 x<3 x-3 x+12$ | $2 x-3 x=3 x-3 x+12$ |  |  |
| $-x<12$ | $-x=12$ | $-12<x$ | $\frac{}{-12}(\bar{\Longrightarrow} x$ |
| $\frac{-x}{-1}>\frac{12}{-1}$ | $\frac{-x}{-1}=\frac{12}{-1}$ |  | $\xrightarrow[-12]{ } x$ |
| $x>-12$ | $x=-12$ | $(-12, \infty)$ |  |

Example 6:
Solve $2 x+3 \leq 3 x+15$.

## Solution:

Inequality

$$
\begin{aligned}
2 x+3 & \leq 3 x+15 \\
2 x+3-3 & \leq 3 x+15-3 \\
2 x & \leq 3 x+12 \\
2 x-3 x & \leq 3 x-3 x+12 \\
-x & \leq 12 \\
\frac{-x}{-1} & \geq \frac{12}{-1} \\
x & \geq-12
\end{aligned}
$$

## Equation

$$
\left.\begin{array}{rl|l}
2 x+3 & =3 x+15 \\
2 x+3-3 & =3 x+15-3 \\
2 x & =3 x+12 \\
2 x-3 x & =3 x-3 x+12 \\
-x & =12 \\
\frac{-x}{-1} & =\frac{12}{-1} \\
x & =-12
\end{array} \right\rvert\, \begin{aligned}
& x \text { not }<-12 \\
& x=\text { or }>-12 \\
& x \text { at least }-12 \\
& {[-12, \infty)}
\end{aligned}
$$

Graph for inequality

### 21.4 Exercise 21

1. Mary Jane wants to apply to Emperor Medical School. She must score at least 90 points average to earn an A in algebra. Her test scores were $\mathbf{8 8}, \mathbf{9 4}, \mathbf{8 9}$, and $\mathbf{9 1}$. At least how many points does she need to get on the fifth and last exam if the grades are weighted equally?
2. Five times the difference of a number and $\mathbf{6}$ is greater than or equal to seven times the number. Find the largest number that satisfies the inequality.
3. Solve $12 x+3<27$.
4. Solve $\mathbf{1 2 x}+\mathbf{3} \leq \mathbf{2 7}$.
5. Solve $2 \boldsymbol{x}-\mathbf{3}<\mathbf{5} \boldsymbol{x}+\mathbf{1 5}$.
6. Solve $\mathbf{2 x}-\mathbf{3} \leq \mathbf{5} \boldsymbol{x}+\mathbf{1 5}$.
7. Solve $\mathbf{9}<\mathbf{2 x}-\mathbf{5} \leq \mathbf{2 5}$.

## STOP!

1. Mary Jane wants to apply to Emperor Medical School. She must score at least $\mathbf{9 0}$ points average to earn an A in algebra. Her test scores were $\mathbf{8 8}, \mathbf{9 4}, \mathbf{8 9}$, and $\mathbf{9 1}$. At least how many points does she need to get on the fifth and last exam if the grades are weighted equally?

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the minimum number of points Mary Jane must score. The total number of points on 4 exams was
$89+94+89+91=363$.
Equation:

$$
\begin{aligned}
\text { Average } & \geq 90 \\
\frac{363+x}{5} & \geq 90 \\
\frac{(363+x)(5)}{5} & \geq(90)(5) \\
363+x & \geq 450 \\
x & \geq 450-363=87
\end{aligned}
$$

Mary Jane must score at least $\mathbf{8 7}$ points on the last exam.
2. Five times the difference of a number and $\mathbf{6}$ is greater than or equal to seven times the number. Find the largest number that satisfies the inequality.

## Solution:

Preamble:
(a) Five times the difference of a number and $\mathbf{6}$ is greater than or equal to seven times the number.
(b) Five times the difference of $\boldsymbol{x}$ and $\mathbf{6}$ is greater than or equal to seven times $\boldsymbol{x}$.
(c) Five times $\boldsymbol{x}-\mathbf{6}$ is greater than or equal to seven times $\boldsymbol{x}$.
(d) $\mathbf{5}(\boldsymbol{x}-\mathbf{6})$ is greater than or equal to $\mathbf{7 x}$.
(e) $5(x-6) \geq 7 x$.

Equation:

$$
\begin{aligned}
5(x-6) & \geq 7 x \\
5 x-30 & \geq 7 x \\
-30 & \geq 7 x-5 x \\
-30 & \geq 2 x \\
-15 & \geq x \\
x & \leq-15
\end{aligned}
$$

$\mathbf{- 1 5}$ is the largest integer that satisfies the original inequality.
3. Solve $\mathbf{1 2 x}+\mathbf{3}<\mathbf{2 7}$.

Solution:

| Inequality | Equation | Interval notation | Graph for inequality |
| :---: | :---: | :---: | :---: |
| $12 x+3<27$ | $12 x+3=27$ |  |  |
| $12 x+3-3<27-3$ | $12 x+3-3=27-3$ |  |  |
| $12 x<24$ | $12 x=24$ | $\boldsymbol{x}$ is less than 2 |  |
| $\underline{12 x}<24$ | $\underline{12 x}=\frac{24}{12}$ |  | ) 2 |
| $\begin{aligned} \overline{12} & <\overline{12} \\ x & <{ }^{2}\end{aligned}$ | $\begin{aligned} \overline{12} & =\overline{12} \\ x & ={ }_{2}\end{aligned}$ | $(-\infty, 2)$ | $\Longrightarrow 2$ |

4. Solve $\mathbf{1 2 x}+\mathbf{3} \leq \mathbf{2 7}$.

## Solution:

| Inequality |  | Equation |  | Interval notation Graph for the inequality |
| :---: | :---: | :---: | :---: | :---: |
| $12 x+3 \leq$ | 27 | $12 x+3=$ | 27 |  |
| $12 x+3-3 \leq$ | 27-3 | $12 x+3-3=$ | 27-3 | $\boldsymbol{x}$ is not more than $\mathbf{2}$ |
| $12 x \leq$ | 24 | $12 x=$ | 24 | $\boldsymbol{x}$ is less than or equal to $\mathbf{2}$ |
| $12 x$ | 24 | $12 x$ | 24 |  |
| 12 | 12 | 12 |  | $\boldsymbol{x}$ is at most 2 |
| $\boldsymbol{x} \leq$ | 2 | $x=$ | 2 | $(-\infty, 2] \quad \text { 号 } 2$ |

5. Solve $\mathbf{2 x}-\mathbf{3}<\mathbf{5} \boldsymbol{x}+\mathbf{1 5}$.

## Solution:

Inequality

$$
\begin{aligned}
2 x-3 & <5 x+15 \\
2 x-3+3 & <5 x+15+3 \\
2 x & <5 x+18 \\
2 x-5 x & <5 x-5 x+18 \\
-3 x & <18 \\
\frac{-3 x}{-3} & >\frac{18}{-3} \\
x & >-6
\end{aligned}
$$

Equation

$$
\begin{array}{rl|ll}
\text { Equation } & & \begin{array}{l}
\text { Int. notation } \\
\text { for inequality }
\end{array} & \text { Graph } \\
2 x-3 & =5 x+15 \\
2 x-3+3 & =5 x+15+3 \\
2 x & =5 x+18 \\
2 x-5 x & =5 x-5 x+18 \\
-3 x & =18 \\
\frac{-3 x}{-3} & =\frac{18}{-3} \\
x & =-6 & &
\end{array}
$$

6. Solve $2 \boldsymbol{x}-\mathbf{3} \leq 5 \boldsymbol{x}+\mathbf{1 5}$.

## Solution:

$$
\begin{aligned}
& \text { Inequality } \\
& \\
& 2 x-3 \leq 5 x+15 \\
& 2 x-3+3 \leq 5 x+15+3 \\
& 2 x \leq 5 x+18 \\
& 2 x-5 x \leq 5 x-5 x+18 \\
&-3 x \leq 18 \\
& \frac{-3 x}{-3} \geq \frac{18}{-3} \\
& x \geq-6
\end{aligned}
$$

7. Solve $\mathbf{9}<2 x-5 \leq 25$.

## Solution:



## Chapter 22

## Plotting Points. Rectangular or Cartesian Coordinate System

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### 22.1 YouTube

https://www.youtube.com/playlist?list=PLBCF15B6E241F7743\&feature=view_all

### 22.2 The Coordinate System

René Descartes is a French mathematician and philosopher. He is reported to have observed a fly crawling on the ceiling while lying in bed. He allegedly wanted to describe the path of the fly. That led him to invent the Cartesian coordinates system. The system is also called the rectangular coordinate system.


The horizontal number line is called the $\boldsymbol{x}$-axis (normally the variable is $\boldsymbol{x}$ ).

The vertical number line is called the $\boldsymbol{y}$-axis (normally the variable is $\boldsymbol{y}$ ).

The point where the two axes intersect is called the origin.

The axes divide the plane containing them (the paper the graph is printed on) into four infinite regions, called quadrants. Traditionally Roman numerals I, II, III and IV are used.

Starting at the origin, you can move to the right in the positive direction. If you move to the left, you travel in the negative direction. Upward from the $\boldsymbol{x}$-axis is positive, downward is negative.


Every point in the Cartesian coordinate plane is labeled with two coordinates. The first coordinate is the $\boldsymbol{x}$-coordinate or abscissa.

The second coordinate is the $\boldsymbol{y}$-coordinate or ordinate. The $\boldsymbol{x}$-value is always the first. That is why we refer to an ordered pair.

Definition of average change of $\boldsymbol{y}$ with respect to $\boldsymbol{x}$.
The average is the change of $\boldsymbol{y}$ divided by the change of $\boldsymbol{x}$. (We shall also call this ratio the slope.)
The rate from point $\boldsymbol{B}(-\mathbf{4}, \mathbf{3})$ to point $\boldsymbol{E}(\mathbf{0}, \mathbf{5})$
is $\frac{5-3}{0-(-4)}=\frac{2}{4}=\frac{1}{2}$.

### 22.3 Examples

Example 1:


Plot the points $\boldsymbol{A}(\mathbf{2}, \mathbf{5}), \boldsymbol{B}(-\mathbf{4}, \mathbf{3})$, $C(-6,-2), D(2,-1), E(0,5)$, and $\boldsymbol{F}(-5,0)$.

## Solution:



Example 2:
Find the coordinates of $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$.

## Solution:

$\boldsymbol{U}(4,2)$ you move right 4 units and up 2 units.
$\boldsymbol{V}(-\mathbf{6}, \mathbf{5})$ you move left $\mathbf{6}$ units and up $\mathbf{5}$ units.
$\boldsymbol{W}(-4,-4)$ you move left 4 units and down 4 units.
$\boldsymbol{X}(0,-4)$ you do not move left or right, but you travel 4 units down.
$\boldsymbol{Y}(5,0)$ you move 5 units right, but do not travel up or down.
$\boldsymbol{Z}(-5,0)$ you move 5 units left, but do not travel up or down.
Example 3:
According to the Dept. of Health and Human Services, MedicAid changed from $\$ \mathbf{7 . 0}$ billion in 1984 to $\mathbf{1 4 . 5}$ billion in 2007. Find the rate if change of dollars per year.

## Solution:

Change in amount: $14.5-7.0=7.5$ billion dollars.
Change in years: $2007-1984=\mathbf{2 3}$ years
Rate of change: $\frac{\mathbf{7 . 5}}{\mathbf{2 3}}=\mathbf{0 . 3 2 6}$ billion dollars per year.

## Example 4:

Carbon monoxide emission from fuel consumption in 1970 was 237 thousand tons. In 2008 the number had risen to 699. Find the average rate in thousand tons per year. Round to the nearest whole number.

## Solution:

Change in tonnage: $699-237=462$ tons
Change in years: $\mathbf{2 0 0 8}-\mathbf{1 9 7 0}=\mathbf{3 8}$ years
Rate of change: $\frac{462}{38}=12$ thousand tons per year.

Example 5:
The rate of change of distance with respect to time is commonly known. What is it if you cover $\mathbf{1 7 0 0}$ miles in 3.5 hours?

## Solution:

The rate (or speed) is $\frac{\mathbf{1 7 0 0} \text { miles }}{\mathbf{3 . 5} \text { hours }}=\mathbf{4 8 5 . 7} \mathrm{mph}$.

### 22.4 Exercise 22

1. Plot the points $\boldsymbol{P}(-3,-4)$,
$Q(3,-5), R(6,2), S(-1,3)$, $\boldsymbol{T}(\mathbf{0}, \mathbf{0})$, and $\boldsymbol{U}(\mathbf{5}, \mathbf{0})$.
2. Find the coordinates of
$\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$.
The points are on the diagram on the right.

3. According to some study, the population in a country changed from $\mathbf{1 2 . 0}$ billion in 1960 to 11.5 billion in 2010. Find the rate if change of people per year.
4. Carbon monoxide emission from industrial processes in 1970 was $\mathbf{1 0}, \mathbf{6 1 0}$ thousand tons. In 2008 the number had fallen to $\mathbf{3 , 2 8 3}$ thousand tons. Find the average rate in thousand tons per year. Round to the nearest whole number.
5. The rate of change of distance is commonly known. What is it if you cover $\mathbf{2 , 4 7 5}$ miles in 4.5 hours?

## STOP!

1. Plot the points
$P(-3,-4)$,
$Q(3,-5)$,
$R(6,2)$,
$S(-1,3)$,
$T(0,0)$, and
$U(5,0)$.

## Solution: <br> 2. Find the coordinates of $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$. The points are on the diagram on the right.

## Solution:


$C(-1,-3)$ because you move left 1 units and down $\mathbf{3}$ units.
$\boldsymbol{D}(-4,0)$ because you move left 4 units and no units up or down.
$\boldsymbol{E}(\mathbf{1}, \mathbf{1})$ because you move right $\mathbf{1}$ unit and up $\mathbf{1}$ unit.
$\boldsymbol{F}(-4,1)$ because you move left 4 units and 1 unit up.
$\boldsymbol{G}(4,5)$ because you move 4 units right and 5 units up.
$\boldsymbol{H}(-4,-5)$ because you move 4 units left and 5 units down.
3. According to some study, the population in a country changed from $\mathbf{1 2 . 0}$ billion in 1960 to 11.5 billion in 2010. Find the rate if change of people per year.
Solution:
Change in population: $\mathbf{1 1 . 5} \mathbf{- 1 2 . 0}=\mathbf{- 0 . 5}$ people.
Change in years: $\mathbf{2 0 1 0}-\mathbf{1 9 6 0}=\mathbf{5 0}$ years
Rate of change: $\frac{\mathbf{- 0 . 5}}{\mathbf{5 0}}=\mathbf{- 0 . 0 1}$ billion people per year.
4. Carbon monoxide emission from industrial processes in 1970 was $\mathbf{1 0}, \mathbf{6 1 0}$ thousand tons. In 2008 the number had fallen to $\mathbf{3 , 2 8 3}$. Find the average rate in thousand tons per year. Round to the nearest whole number.

## Solution:

Change in tonnage: $\mathbf{3 , 2 8 3 - 1 0 , 6 1 0}=-\mathbf{7 , 3 2 7}$ tons
Change in years: $\mathbf{2 0 0 8} \mathbf{- 1 9 7 0}=\mathbf{3 8}$ years
Rate of change: $\frac{-\mathbf{7 , 3 2 7}}{\mathbf{3 8}}=-\mathbf{1 9 3}$ thousand tons per year.
5. The rate of change of distance is commonly known. What is it if you cover $\mathbf{2 , 4 7 5}$ miles in $\mathbf{4 . 5}$ hours?

Solution:
The rate (or speed) is $\frac{\mathbf{2 , 4 7 5} \text { miles }}{4.5 \text { hours }}=\mathbf{5 5 0} \mathrm{mph}$.

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## Chapter 23

## Graph Linear Equations. Intercepts

### 23.1 YouTube

https://www.youtube.com/playlist?list=PL4C0DCC59200C049F\&feature=view_all

### 23.2 Basics

Is $\boldsymbol{B}(\mathbf{3}, \mathbf{- 2})$ a solution of $2 x+3 y=12$ ? What are you asked to do? You are asked to evaluate $\mathbf{2 x}+\mathbf{3 y}=\mathbf{1 2}$ if $\boldsymbol{x}=\mathbf{3}$ and $\boldsymbol{y}=\mathbf{- 2}$.

$$
\begin{aligned}
2 x+3 y & =12 \\
2()+3() & =12 \\
2(3)+3(-2) & =12 \\
6-6 & =12 \quad \text { No, } B(3,-2) \text { isn't } \\
0 & =12 \quad \text { a solution of } 2 x+3 y=12
\end{aligned}
$$

Is $\boldsymbol{A}(\mathbf{3}, \mathbf{2})$ a solution of $\mathbf{2 x + 3} \boldsymbol{y}=12$ ? Eval-
 uate $2 x+3 y=12$ if $x=2$ and $y=3$.

$$
\begin{aligned}
2 x+3 y & =12 \\
2()+3() & =12 \\
2(3)+3(2) & =12 \text { Yes, } A(3,2) \text { is } \\
12 & =12 \quad \text { a solution of } 2 x+3 y=12
\end{aligned}
$$

How many lines can you draw through $\boldsymbol{A}(\mathbf{3}, \mathbf{2})$. Take a look. As a matter of fact, infinitely many.
Find another point on the line. Let $\boldsymbol{x}$ be any number, like $\boldsymbol{x}=\mathbf{0}$. Solve

$$
\begin{aligned}
2 x+3 y & =12 \\
2()+3 y & =12 \\
2(0)+3 y & =12 \\
0+3 y & =12 \\
\frac{3 y}{3} & =\frac{12}{3} \\
y & =4 \quad C(0,4) \text { is another point that satisfies the equation } 2 x+3 y=12
\end{aligned}
$$

Not only is $\boldsymbol{C}(\mathbf{0}, \mathbf{4})$ another point that satisfies the equation $\mathbf{2 x}+\mathbf{3 y}=\mathbf{1 2}$, it identifies one (and only one) line that passes through $\boldsymbol{A}$ and $\boldsymbol{C}$.
$\boldsymbol{C}$ is special. It is where the line intersects the vertical line $\boldsymbol{x}=\mathbf{0}$ or $\boldsymbol{y}$-axis. $\boldsymbol{C}$ is called the $\boldsymbol{y}$-intercept. Note that all the points on the $\boldsymbol{y}$-axis have abscissas ( $\boldsymbol{x}$-values) $\boldsymbol{x}=\mathbf{0}$. Points do not move right or left.

The graph of the line drawn using $2 \boldsymbol{x}+\mathbf{3 y}=\mathbf{1 2}$ must have an $\boldsymbol{x}$-intercept. The $\boldsymbol{x}$-intercepts have ordinates $(\boldsymbol{y}$-value) $\boldsymbol{y}=\mathbf{0}$. From the graph the $\boldsymbol{x}$-intercept seems to be (6,0). Is that correct mathematically? Points do not move up or down.

Graphing a line given its equation boils down to finding two points on the line. (Most authors suggest finding a third point merely for checking that the three points are collinear (belong to the same line).)

Any equation in the form $\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{y}=\boldsymbol{C}$ has a graph which is a line. That's why it is called a linear equation. The exponents of $\boldsymbol{x}$ and $\boldsymbol{y}$ are 1. $\boldsymbol{A}, \boldsymbol{B}$. and $\boldsymbol{C}$ are constants. $\boldsymbol{A} \boldsymbol{B}$ are not both zero.

### 23.3 Examples

Example 1:
Is $\boldsymbol{A}(4,3)$ a point on the line $\mathbf{3 x}-4 \boldsymbol{y}=12$ ?

## Solution:

$$
\begin{aligned}
3 x-4 y & =12 \\
3()-4() & =12 \\
3(4)-4(3) & =12 \\
12-12 & =12 \\
0 & =12 \text { No, } A(4,3) \text { is not a point on the line } 3 x-4 y=12
\end{aligned}
$$

Example 2:
Find the intercepts of the line (whose equation is) $\mathbf{3 x}-\mathbf{4 y}=\mathbf{1 2}$.

## Solution:

For the $\boldsymbol{x}$-intercept we need the ordinate to be $\boldsymbol{y}=\mathbf{0}$. Then

$$
\begin{aligned}
3 x-4 y & =12 \\
3 x-4(0) & =12 \\
3 x & =12 \\
x & =\frac{12}{3}=4 \quad A(4,0) \text { is the } x \text {-intercept. }
\end{aligned}
$$

For the $\boldsymbol{y}$-intercept we need the abscissa to be $\boldsymbol{x}=\mathbf{0}$. Then

$$
\begin{aligned}
3 x-4 y & =12 \\
3()-4 y & =12 \\
3(0)-4 y & =12 \\
y & =\frac{12}{-4}=-3
\end{aligned}
$$

$\boldsymbol{B}(\mathbf{0}, \mathbf{- 3})$ is the $\boldsymbol{y}$-intercept.

## Example 3:

Graph the line $\mathbf{3 x}-\mathbf{4 y}=\mathbf{1 2}$ using the intercepts.

## Solution:

$\boldsymbol{A}(\mathbf{4}, \mathbf{0})$ and $\boldsymbol{B}(\mathbf{0},-\mathbf{3})$ are the intercepts.
Example 4:
Sketch the graph of $\boldsymbol{y}=\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}+\mathbf{2}$

## Solution:

Finding the $\boldsymbol{y}$-intercept is easy. Just substitute $\boldsymbol{x}=\mathbf{0}$ into the equation to get $\boldsymbol{y}=\mathbf{2}$.

In this example, use a multiple of $\mathbf{3}$ for $\boldsymbol{x}$, like $\boldsymbol{x}=\mathbf{3}$ or $\boldsymbol{x}=\mathbf{6}$. We avoid getting into fractions.

$$
\begin{aligned}
y & =\frac{1}{3} x+2 \\
y & =\frac{1}{3}()+2 \\
y & =\frac{1}{3}(3)+2 \\
y & =1+2 \\
y & =3
\end{aligned}
$$



The line we seek passes through $\boldsymbol{C}(\mathbf{0}, \mathbf{2})$ and $\boldsymbol{D}(\mathbf{3}, \mathbf{3})$.
Bonus: The graph shows that the $\boldsymbol{y}$-intercept is $\boldsymbol{E}(-\mathbf{6}, \mathbf{0})$.

Example 5:
Find two points on the line $\frac{\mathbf{2}}{\mathbf{3}} \boldsymbol{x}+\frac{\mathbf{4}}{\mathbf{5}} \boldsymbol{y}=\mathbf{8}$. Try to get integer solutions.

## Solution:

You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.

$$
\begin{aligned}
\frac{2}{3}(0)+\frac{4}{5} y & =8 \\
\frac{4}{5} y & =1 \\
\frac{5}{4} \cdot \frac{4}{5} y & =\frac{5}{4} \\
y & =\frac{5}{4} \text { not an integer. }
\end{aligned}
$$

If $\boldsymbol{x}=\mathbf{0}$ then

Use multiples of the LCD of the denominators (like $\boldsymbol{x}=\mathbf{1 5 m}$ with $\boldsymbol{m}=\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots$ ). Let $\boldsymbol{x}=\mathbf{3 0}$, then

$$
\begin{aligned}
\frac{2}{3}()+\frac{4}{5} y & =8 \\
\frac{2}{3}(30)+\frac{4}{5} y & =8 \\
2 \cdot 10+\frac{4}{5} y & =8 \\
\frac{4}{5} y & =8-20 \\
\frac{4}{5} y & =-12 \\
\frac{5}{4} \cdot \frac{4}{5} y & =-\frac{5}{4}(12) \\
y & =-5 \cdot 3=-15 \quad(30,-15) \text { is a point on the line. }
\end{aligned}
$$

Let $\boldsymbol{x}=\mathbf{- 1 2}$, (which happens to work in spite of not being a multiple of $\mathbf{1 5}$, ) then

$$
\begin{aligned}
\frac{2}{3}()+\frac{4}{5} y & =8 \\
\frac{2}{3}(-12)+\frac{4}{5} y & =8 \\
(-8)+\frac{4}{5} y & =8 \\
\frac{4}{5} y & =16 \\
\frac{4}{5} y & =16 \\
\frac{5}{4} \cdot \frac{4}{5} y & =20 \quad(-12,20) \text { is a point on the line. }
\end{aligned}
$$

Can you find additional points with integer coordinates?
Example 6:
$\boldsymbol{x}=\mathbf{3}$ is an equation whose graph is a line. Find two points on this line and graph the line.

## Solution:

A point has two coordinates. $\boldsymbol{A}(\mathbf{3}, \mathbf{1})$ is such a point. Why is $A(3,1)$ a solution of $\boldsymbol{x}=\mathbf{3}$ ? Where do we substitute $\boldsymbol{y}=\mathbf{1}$ ?

Think of $\boldsymbol{x}=\mathbf{3}$ as $\boldsymbol{x}+\mathbf{0}=\mathbf{3}$ or better $x+0 y=3$.

Then $\boldsymbol{A}(\mathbf{3}, \mathbf{1})$ is a solution because

$$
\begin{aligned}
()+0() & =3 \\
(3)+0(1) & =3 \\
3+0 & =3
\end{aligned}
$$

The $\boldsymbol{y}$-term will always drop out because it is multiplied by $\mathbf{0}$.
$\boldsymbol{A}(\mathbf{3}, \mathbf{1}), \boldsymbol{A}(\mathbf{3}, \mathbf{7}), \boldsymbol{A}(\mathbf{3},-\boldsymbol{\pi}), \ldots$ are all
 points on the vertical line $\boldsymbol{x}=\mathbf{3}$.
Example 7:
Find the intercepts of $\frac{\mathbf{1}}{\mathbf{5}} \boldsymbol{x}+\frac{\mathbf{1}}{\mathbf{7}} \boldsymbol{y}=\mathbf{1}$.

## Solution:

If $\boldsymbol{x}=\mathbf{0}, \frac{\mathbf{1}}{\mathbf{5}}(\mathbf{0})+\frac{\mathbf{1}}{\mathbf{7}} \boldsymbol{y}=\mathbf{1}$ or $\frac{\mathbf{7}}{\mathbf{7}} \boldsymbol{y}=\mathbf{7}$ so that $(\mathbf{0}, \mathbf{7})$ is the $\boldsymbol{y}$-intercept.
If $y=0, \frac{1}{\mathbf{5}} x+\frac{\mathbf{1}}{\mathbf{7}}(0)=1$ or $\frac{\mathbf{5}}{\mathbf{5}} \boldsymbol{y}=\mathbf{5}$ so that $(5,0)$ is the $\boldsymbol{x}$-intercept.
In general, if $\frac{\boldsymbol{x}}{\boldsymbol{a}}+\frac{\boldsymbol{y}}{\boldsymbol{b}}=\mathbf{1}$, the intercepts are $(\mathbf{0}, \boldsymbol{b})$ and $(\boldsymbol{a}, \mathbf{0})$.

### 23.4 Exercise 23

1. Is $\boldsymbol{A}(\mathbf{5},-2)$ a point on the line $\mathbf{2 x}-\mathbf{5} \boldsymbol{y}=\mathbf{2 0}$ ?
2. Find the intercepts of the line (whose equation is)

$$
5 x-3 y=6
$$

3. Graph the line $\mathbf{2 x}+\mathbf{3} \boldsymbol{y}=\mathbf{6}$ using the intercepts.
4. Sketch the graph of $\boldsymbol{y}=\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}+\mathbf{2}$
5. Find two points on the line $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}+\frac{\mathbf{3}}{\mathbf{4}} \boldsymbol{y}=\mathbf{1 0}$. Try to get integer solutions.
6. $\boldsymbol{y}=-\mathbf{2}$ is an equation whose graph is a line. Find two points on this line and graph the line.
7. Find the intercepts of $\frac{\mathbf{1}}{\mathbf{9}} \boldsymbol{x}-\frac{\mathbf{1}}{\mathbf{5}} \boldsymbol{y}=\mathbf{1}$.

## STOP!

1. Is $\boldsymbol{A}(\mathbf{5},-\mathbf{2})$ a point on the line $\mathbf{2 x}-\mathbf{5} \boldsymbol{y}=\mathbf{2 0}$ ?

Solution:

$$
\begin{aligned}
& 2 x-5 y=20 \\
& 2()-5()=20 \\
& 2(5)-5(-2)=20 \\
& 10+10=20 \quad \text { Yes, } A(5,-2) \text { is a point } \\
& \\
& \quad \text { on the line } 2 x-5 y=20 .
\end{aligned}
$$

2. Find the intercepts of the line (whose equation is)

$$
5 x-3 y=6
$$

## Solution:

For the $\boldsymbol{x}$-intercept we need the ordinate to be $\boldsymbol{y}=\mathbf{0}$. Then

$$
\begin{aligned}
5 x-3 y & =6 \\
5 x-3(0) & =6 \\
5 x & =6 \\
x & =\frac{6}{5} A\left(\frac{6}{5}, 0\right) \text { is the } x \text {-intercept. }
\end{aligned}
$$

For the $\boldsymbol{y}$-intercept we need the abscissa to be $\boldsymbol{x}=\mathbf{0}$. Then

$$
\begin{aligned}
5 x-3 y & =6 \\
5(0)-3 y & =6 \\
-3 y & =6 \\
y & =\frac{6}{-3}=-2 \quad B(0,-2) \text { is the } y \text {-intercept. }
\end{aligned}
$$



To avoid fractions for $\boldsymbol{y}$ substitute multiples of $\mathbf{3}$ in place of $\boldsymbol{x}$.

$$
\begin{aligned}
y & =\frac{1}{3} x+2 \\
y & =\frac{1}{3}(6)+2 \\
y & =\frac{1}{3}(6)+2 \\
y & =2+2 \\
y & =4
\end{aligned}
$$

The line we seek passes through $\boldsymbol{B}(\mathbf{0}, \mathbf{2})$ and $\boldsymbol{C}(\mathbf{6}, \mathbf{4})$.
Bonus: The graph shows that the $\boldsymbol{y}$-intercept is $\boldsymbol{E}(\mathbf{- 6 , 0})$.
5. Find two points on the line $\frac{\mathbf{1}}{\mathbf{3}} \boldsymbol{x}+\frac{\mathbf{3}}{\mathbf{4}} \boldsymbol{y}=\mathbf{1 0}$. Try to get integer solutions.

## Solution:

You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.

$$
\frac{1}{3}(0)+\frac{3}{4} y=10
$$

If $\boldsymbol{x}=\mathbf{0}$ then $\quad \frac{3}{4} \boldsymbol{y}=\mathbf{1 0}$

$$
\frac{4}{3} \cdot \frac{3}{4} y=\frac{40}{3} \quad \text { not an integer. }
$$

Use multiples of the LCD of the denominators (like $\boldsymbol{x}=12$ ).

$$
\begin{aligned}
\frac{1}{3}()+\frac{3}{4} y & =10 \\
\frac{1}{3}(12)+\frac{3}{4} y & =10 \\
4+\frac{3}{4} y & =10 \\
\frac{3}{4} y & =10-4 \\
\frac{3}{4} y & =6 \\
\frac{4}{3} \cdot \frac{3}{4} y & =\frac{4}{3}(6) \\
y & =4 \cdot 2 \\
y & =8
\end{aligned}
$$

Let $x=48$.

$$
\begin{aligned}
\frac{1}{3}()+\frac{3}{4} y & =10 \\
\frac{1}{3}(48)+\frac{3}{4} y & =10 \\
16+\frac{3}{4} y & =10 \\
\frac{3}{4} y & =10-16 \\
\frac{3}{4} y & =-6 \\
\frac{4}{3} \cdot \frac{3}{4} y & =\frac{4}{3}(-6) \\
y & =4 \cdot(-2) \\
y & =-8
\end{aligned}
$$

$\boldsymbol{x}=\mathbf{2 4}$ and $\boldsymbol{x}=\mathbf{3 6}$ did not lead to integers. Can you find additional points with integer coordinates?
6. $\boldsymbol{y}=-\mathbf{2}$ is an equation whose graph is a line. Find two points on this line and graph the line.

## Solution:

A point has two coordinates. $\boldsymbol{A}(\mathbf{3}, \mathbf{- 2 )}$ is such a point. Why is $\boldsymbol{A}(\mathbf{3},-2)$ a solution of $\boldsymbol{y}=\mathbf{- 2}$ ? Where do we substitute $x=3$ ?

Think of $\boldsymbol{y}=\mathbf{- 2}$ as $\mathbf{0}+\boldsymbol{y}=\mathbf{- 2}$ or better $\mathbf{0} \boldsymbol{x}+\boldsymbol{y}=\mathbf{- 2}$.

Then $\boldsymbol{A}(\mathbf{3}, \mathbf{- 2 )}$ is a solution

$$
\begin{aligned}
0(3)+y & =-2 \\
0+(-2) & =-2 \\
-2 & =-2
\end{aligned}
$$

The $\boldsymbol{x}$-term will always drop out.It is multiplied by $\mathbf{0}$.
$(3,-2),(7,-2),(-\pi,-2), \ldots$ are all points on the horizontal line $\boldsymbol{y}=\mathbf{- 2}$.
7. Find the intercepts of $\frac{1}{9} x-\frac{1}{5} y=1$.


## Solution:

If $x=0, \frac{1}{9}(0)-\frac{1}{5} y=1$ or $\frac{1}{-5} y=1$ so that $(0,-5)$ is the $\boldsymbol{y}$-intercept.
If $y=0, \frac{1}{\mathbf{9}} x-\frac{\mathbf{1}}{\mathbf{5}}(0)=1$ or $\frac{\mathbf{1}}{\mathbf{9}} \boldsymbol{x}=1$ so that $(\mathbf{9}, 0)$ is the $\boldsymbol{x}$-intercept.

## Chapter 24

## Rates/Slopes

### 24.1 YouTube

### 24.2 Basics

Plot the points with coordinates $A(-5,-3),, \quad B(-3,-1), \quad C(0,2)$, $\boldsymbol{D}(\mathbf{2}, 4), \boldsymbol{E}(\mathbf{3}, 5)$. They seem to fall on a straight line.

Did you note that the ordinate ( $\boldsymbol{y}$-value) is always 2 units more than the abscissa ( $x$-value)?


In general, then, $\boldsymbol{y}=\boldsymbol{x}+\mathbf{2} . \boldsymbol{x}$ can be any real number.

Consider points $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{1 . 1}, \mathbf{3 . 1})$ on the line. $(1.05,3.05)$ has abscissa $x=1.05$ midway between $\boldsymbol{x}=\mathbf{1 . 0}$ and 1.1.
$(1.025,3.025)$ has abscissa $\boldsymbol{x}=1.025$ midway between $\boldsymbol{x}=\mathbf{1 . 0}$ and $\mathbf{1 . 0 5}$.

We can repeat this process over and over again, forever. Thus we can identify and plot infinitely many points with rational as well as irrational numbers for the abscissa. Every real number $\boldsymbol{x}$ has a corresponding number $\boldsymbol{y}=\boldsymbol{x}+\mathbf{2}$. Stringing these infinitely many points together leads to a line.


More and more points bunch together as we plot the midpoint between $(\mathbf{1}, \mathbf{0})$ and the previous point.

### 24.3 Rates/Slopes. Examples

Example 1:
Find the missing coordinate for the points on the line whose equation is $\mathbf{5 x}+\mathbf{2 y}=\mathbf{9}$.

| $x$ | 0 |  | 4 |  | $\frac{2}{5}$ |  |
| :--- | :--- | :--- | :--- | :--- | :---: | :---: |
| $y$ |  | 0 |  | 5 |  | $\frac{5}{2}$ |

Solution:
$x=0$, so $5(0)+2 y=9$ or $y=\frac{9}{2}$
$y=0$, so $5 x+2(0)=9$ or $x=\frac{\mathbf{9}}{\mathbf{5}}$
$x=4$, so $5(4)+2 y=9,2 y=9-20, \quad 2 y=-11$, or $y=\frac{-11}{2}$
$y=5$, so $5 x+2(5)=9,5 x=9-10$, or $x=\frac{-1}{5}$
$x=\frac{2}{5}, 5\left(\frac{2}{5}\right)+2 y=9,2 y=9-2,2 y=7$, or $y=\frac{7}{2}$
$y=\frac{5}{2}$, so $5 x+2\left(\frac{5}{2}\right)=9,5 x=9-5$, or $x=\frac{4}{5}$

| $x$ | 0 | $\frac{9}{5}$ | 4 | $\frac{-1}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | $\frac{9}{2}$ | 0 | $\frac{-11}{2}$ | 5 | $\frac{7}{2}$ | $\frac{5}{2}$ |

A line can change direction. One way of quantifying direction or inclination is the concept of rate. You are familiar with this concept already. What is the legal maximum rate of change of distance with respect to time on California freeways?

What is what? You never heard of speed? Yes, speed is a rate, specifically $65 \frac{\text { miles }}{\text { hour }}=\frac{\text { Distance }}{\text { Time }}$.

Let's illustrate the concept of rate using a Cartesian coordi-
 nate system.

Let's go back to René Descartes' fly on the ceiling. The fly crawled from $\boldsymbol{A}(\mathbf{0}, \mathbf{1})$ to $\boldsymbol{B}(\mathbf{5}, \mathbf{3})$. It crawled 2 feet in 5 seconds. Its rate is $\boldsymbol{R}=\frac{\mathbf{3}-\mathbf{1} \mathrm{ft}}{\mathbf{5}-\mathbf{0} \mathrm{sec}}$.

Imagine an infinite line passing through $\boldsymbol{A}(\mathbf{0}, \mathbf{1})$ and $\boldsymbol{B}(\mathbf{5}, \mathbf{3})$. We shall give a new name to rate. From now on, we'll talk about a slope of a line.

The letter $\boldsymbol{m}$ is traditionally used for slope (although it is not legally binding).


$$
\boldsymbol{m}=\frac{\text { vertical change in } \boldsymbol{y}}{\text { horizontal change in } \boldsymbol{x}}=\frac{\boldsymbol{y}_{2}-\boldsymbol{y}_{\mathbf{1}}}{\boldsymbol{x}_{\mathbf{2}}-\boldsymbol{x}_{\mathbf{1}}}
$$

Note: Subscripts are not exponents. $\boldsymbol{x}_{\boldsymbol{2}}$ is different than $\boldsymbol{x}^{\mathbf{2}}$. Please be very careful.
Example 2:
Find the slope of the line passing through $\boldsymbol{A}(\mathbf{2}, \mathbf{4})$ and $\boldsymbol{B}(\mathbf{6}, \mathbf{1 0})$.

## Solution:

Let $\left(x_{1}, y_{1}\right)=\boldsymbol{A}(2,4)$ and $\left(x_{2}, y_{2}\right)=B(6,10)$
then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{10-4}{6-2}=\frac{6}{4}=\frac{3}{2}$


Let $\left(x_{1}, y_{1}\right)=\boldsymbol{B}(\mathbf{6}, 10)$ and $\left(x_{2}, y_{2}\right)=\boldsymbol{A}(2,4)$
then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-10}{2-6}=\frac{-6}{-4}=\frac{3}{2}$

But be careful
$\underset{\text { itive. }}{m \neq \frac{10-4}{2-6}=\frac{6}{-4}=-\frac{3}{2}}$
Note that the slope of a rising line is positive.

The slope of a falling line is negative.


Example 3:
Find the slope of the line passing through $\boldsymbol{A}(\mathbf{2}, 4)$ and $\boldsymbol{B}(\mathbf{6},-\mathbf{3})$.

## Solution:

Let $\left(x_{1}, y_{1}\right)=A(2,4)$ and $\left(x_{2}, y_{2}\right)=B(6,-3)$
then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{-3-4}{6-2}=\frac{-7}{4}=-\frac{7}{4}$
Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{B}(\mathbf{6},-\mathbf{3})$ and $\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)=\boldsymbol{A}(\mathbf{2}, \mathbf{4})$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-(-3)}{2-6}=\frac{7}{-4}=-\frac{7}{4}$
Example 4:
Find the slope of the line passing through $\boldsymbol{A}(\mathbf{2}, 4)$ and $\boldsymbol{B}(\mathbf{6}, \mathbf{4})$.
Solution:
Let $\left(x_{1}, y_{1}\right)=\boldsymbol{A}(2,4)$ and $\left(x_{2}, y_{2}\right)=B(6,4)$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-4}{6-2}=\frac{0}{4}=0$
Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{B}(\mathbf{6}, 4)$ and $\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)=\boldsymbol{A}(\mathbf{2}, 4)$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-4}{2-6}=\frac{0}{-4}=0$
The slope of a horizontal line is $\boldsymbol{m}=\mathbf{0}$.
Example 5:
Find the slope of the line passing through $\boldsymbol{A}(\mathbf{2}, 4)$ and $\boldsymbol{B ( 2 , 6 )}$.

## Solution:

Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{A}(\mathbf{2}, \mathbf{4})$ and $\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)=\boldsymbol{B}(\mathbf{2}, \mathbf{6})$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{6-4}{2-2}=\frac{2}{0}$ which is undefined.
Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{B}(\mathbf{6}, 4)$ and $\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)=\boldsymbol{A}(\mathbf{2}, 4)$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-6}{2-2}=\frac{-2}{0}$
The slope of a vertical line is undefined. The slope is undefined, not $\mathbf{0}$ is undefined.
Example 6:
The Internal Revenue Service (IRS) uses linear depreciation.
The value of a computer decreases by a fixed amount each year until its net value is zero.
What value of the computer can you claim on your tax return after 4 years if its original value is $\boldsymbol{\$ 1 , 4 0 0}$ and it depreciates at a fixed rate of $\mathbf{\$ 2 0 0}$ per year?

## Solution:



The value of the computer is $\$ \mathbf{6 0 0}$ after $\mathbf{4}$ years.

### 24.4 Exercise 24

1. Find the slope of the line passing through $\boldsymbol{A}(-\mathbf{2}, 4)$ and $\boldsymbol{B}(\mathbf{6},-\mathbf{1 0})$.
2. Find the slope of the line passing through $\boldsymbol{A}(-\mathbf{2}, \mathbf{4})$ and $\boldsymbol{B}(-\mathbf{6},-\mathbf{3})$.
3. Find the slope of the line passing through $\boldsymbol{A}(\mathbf{3}, 7)$ and $\boldsymbol{B}(-3,7)$.
4. Find the slope of the line passing through $\boldsymbol{A}(-2,4)$ and $\boldsymbol{B}(-2,-4)$.
5. A train leaves a station at $\mathbf{8}: \mathbf{0 0}$ A.M. and travels at an average of $\frac{\mathbf{3 0} \text { miles }}{\text { hour }}$. Approximately at what time will the train be $\mathbf{1 0 0}$ miles from the station?

## STOP!

1. Find the slope of the line passing through $\boldsymbol{A}(-\mathbf{2}, \mathbf{4})$ and $\boldsymbol{B}(\mathbf{6}, \mathbf{- 1 0})$.

## Solution:

Let $\left(x_{1}, y_{1}\right)=\boldsymbol{A}(-\mathbf{2}, 4)$ and $\left(x_{2}, y_{2}\right)=\boldsymbol{B}(\mathbf{6},-\mathbf{1 0})$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{-10-4}{6-(-2)}=\frac{-14}{8}=-\frac{7}{4}$
Let $\left(x_{1}, y_{1}\right)=\boldsymbol{B}(\mathbf{6},-10)$ and $\left(x_{2}, y_{2}\right)=\boldsymbol{A}(-\mathbf{2}, 4)$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-(-10)}{-2-6}=\frac{14}{-8}=-\frac{7}{4}$
2. Find the slope of the line passing through $\boldsymbol{A}(-\mathbf{2}, \mathbf{4})$ and $\boldsymbol{B}(-\mathbf{6},-\mathbf{3})$.

## Solution:

Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{A}(-\mathbf{2}, \mathbf{4})$ and $\left(\boldsymbol{x}_{\mathbf{2}}, \boldsymbol{y}_{2}\right)=\boldsymbol{B}(-\mathbf{6},-\mathbf{3})$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{-3-4}{-6-(-2)}=\frac{-7}{-4}=\frac{7}{4}$
Let $\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)=\boldsymbol{B}(-\mathbf{6},-\mathbf{3})$ and $\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)=\boldsymbol{A}(-\mathbf{2}, \mathbf{4})$ then
$m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-(-3)}{-2-(-6)}=\frac{7}{4}=\frac{7}{4}$
3. Find the slope of the line passing through $\boldsymbol{A}(\mathbf{3}, 7)$ and $\boldsymbol{B}(-\mathbf{3}, 7)$.

Solution:
Let $\left(x_{1}, y_{1}\right)=A(3,7)$ and $\left(x_{2}, y_{2}\right)=B(-3,7)$ then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{7-7}{-3-3}=\frac{0}{-6}=0$
Let $\left(x_{1}, y_{1}\right)=B(-3,7)$ and $\left(x_{2}, y_{2}\right)=A(3,7)$ then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{7-7}{3-(-3)}=\frac{0}{6}=0$
4. Find the slope of the line passing through $\boldsymbol{A}(\mathbf{- 2}, 4)$ and $\boldsymbol{B}(-\mathbf{2}, \mathbf{4})$.

Solution:
Let $\left(x_{1}, y_{1}\right)=A(-2,4)$ and $\left(x_{2}, y_{2}\right)=B(-2,-4)$ then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{-4-4}{-2-(-2)}=\frac{-8}{0}$ which is undefined.
Let $\left(x_{1}, y_{1}\right)=B(-2,-4)$ and $\left(x_{2}, y_{2}\right)=A(-2,4)$ then $m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{4-(-4)}{-2-(-2)}=\frac{8}{0}$
5. A train leaves a station at 8:00 A.M. and travels at an average of $\frac{\mathbf{3 0} \text { miles }}{\text { hour }}$. Approximately at what time will the train be $\mathbf{1 0 0}$ miles from the station?

## Solution:

The train will have traveled approximately 3.2 or 3.3 hours after leaving the station. The time will be about 11:15 A.M. (Approximating is the best we can do graphically. We'll come up with 11:20 A.M. through algebraic methods later.)


## Chapter 25

## Equations of a Line

### 25.1 YouTube

https://www.youtube.com/playlist?list=PL6739B06EFA53C789\&feature=view_all

### 25.2 Basics

$\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{y}=\boldsymbol{C}$ is the standard or general form of the equation of a line. $\boldsymbol{A}$ and $\boldsymbol{B}$ cannot be 0 simultaneously.

Two very useful forms of the equation of a line are the point-slope form and the slope-intercept form. We can derive these from the formula defining the slope of a line passing through two points $\boldsymbol{A}\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)$ and $B\left(x_{2}, y_{2}\right)$,

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

Instead of using a fixed point $\boldsymbol{B}\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{\boldsymbol{2}}\right)$ let's use a running point (any point) on the line $(\boldsymbol{x}, \boldsymbol{y})$.

$$
\begin{array}{rlrl}
\boldsymbol{m} & =\frac{y-y_{1}}{x-x_{1}} & & \frac{\text { change in } \boldsymbol{y}}{\text { change in } \boldsymbol{x}} \\
\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{1}\right) & =\boldsymbol{y}-\boldsymbol{y}_{1} & & \text { multiply both sides by } \boldsymbol{x}-\boldsymbol{x}_{1} \\
\boldsymbol{m} \boldsymbol{x}-\boldsymbol{m} \boldsymbol{x}_{1} & =\boldsymbol{y}-\boldsymbol{y}_{1} & & \text { Point-slope form. } \\
\boldsymbol{m x}-\boldsymbol{m} \boldsymbol{x}_{1}+\boldsymbol{y}_{1} & =\boldsymbol{y} & & \text { Distribute } \boldsymbol{m} \\
\boldsymbol{y} & =\boldsymbol{m x}-\boldsymbol{m} \boldsymbol{x}_{1}+\boldsymbol{y}_{1} & & \text { Add } \boldsymbol{y}_{1} \text { to both sides. } \\
\boldsymbol{y} & =\boldsymbol{m x}+\boldsymbol{b} & & -\boldsymbol{m} \boldsymbol{x}_{1}+\boldsymbol{y}_{1} \text { is a constant, like } \boldsymbol{b} \\
& & \text { Slope-intercept form. }
\end{array}
$$

$\boldsymbol{y}=\boldsymbol{b}$ or $(\mathbf{0}, \boldsymbol{b})$ is the $\boldsymbol{y}$-intercept.

Point-slope form of the equation of a line.

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

Slope-intercept form of the equation of a line.

$$
y=m x+b
$$

$\frac{\boldsymbol{x}}{\boldsymbol{a}}+\frac{\boldsymbol{y}}{\boldsymbol{b}}=\mathbf{1}$ is the double intercept form of the equation of a line.
Both $\boldsymbol{x}$-intercept $(\boldsymbol{a}, \mathbf{0})$ and $\boldsymbol{y}$-intercept $(\mathbf{0}, \boldsymbol{b})$ satisfy the double intercept form of the equation of a line.

### 25.3 Parallel and Perpendicular Lines

Do you remember the property of ratios from similar triangles? Two triangles are similar if they have congruent (equal) angles.


$$
\frac{a}{d}=\frac{b}{e}=\frac{c}{f}
$$

Lines $\boldsymbol{A B}$ and $\boldsymbol{D E}$ are parallel because they form the same angle $\mathbf{3 0}^{\circ}$ with a third line $\boldsymbol{A F}$. Now let's apply the concept to parallel lines.
They have the same slope
$\frac{\overline{B C}}{\overline{A C}}=\frac{\overline{E F}}{\overline{D F}}$
since triangles $\boldsymbol{A C B}$ and $\boldsymbol{D F E}$ are similar.


$$
\begin{aligned}
& \text { Two lines } \boldsymbol{L}_{\mathbf{1}} \text { and } \boldsymbol{L}_{\mathbf{2}} \text { are parallel if (and only if) } \\
& \text { their slopes are equal } \\
& \qquad \boldsymbol{m}_{\mathbf{1}}=\boldsymbol{m}_{\mathbf{2}}
\end{aligned}
$$

Now let's investigate perpendicular lines.
Consider a line drawn on a regular $\mathbf{8} \frac{\mathbf{1}}{\mathbf{2}} \times \mathbf{1 1}$ sheet of paper on a table. We next rotate the sheet of paper $\mathbf{9 0}{ }^{\circ}$. The pre-rotation line is perpendicular to the post-rotation line.


By algebra $m_{1}=\frac{2}{5}=\frac{1}{\frac{5}{2}}=-\frac{1}{-\frac{5}{2}}=-\frac{1}{m_{2}}$
Another way of stating the relation between the slopes is

$$
\left(\frac{2}{5}\right)\left(-\frac{5}{2}\right)=-1 \text { or } m_{1} m_{2}=-1
$$

Two lines $\boldsymbol{L}_{\mathbf{1}}$ and $\boldsymbol{L}_{\mathbf{2}}$ are perpendicular if (and only if) their slopes are negative reciprocals.

$$
m_{1}=-\frac{1}{m_{2}} \quad \text { or } \quad m_{1} m_{2}=-1
$$

### 25.4 Examples

## Example 1:

Find the equation of a line through $\boldsymbol{A}(\mathbf{0}, \mathbf{- 3})$ if the slope is $\boldsymbol{m}=\mathbf{2}$.

## Solution:

Note that $\boldsymbol{A}(\mathbf{0},-\mathbf{3})$ is the $\boldsymbol{y}$-intercept, so $\boldsymbol{b}=-\mathbf{3}$ in the slope-intercept form $\boldsymbol{y}=\mathbf{2 x}-\mathbf{3}$.
Could you have used the point-slope form since you know one point and the slope?
$y-y_{1}=\boldsymbol{m}\left(x-x_{1}\right)$ can be rewritten $y-(-3)=2(x-0)$ or $y=2 x-3$.
Example 2:
Find the equation of the line passing through $\boldsymbol{A}(\mathbf{2},-\mathbf{3})$ and $\boldsymbol{B}(5,3)$.

## Solution:

It would be nice to know the slope.
Well, $\boldsymbol{m}=\frac{\mathbf{3 - ( - 3 )}}{\mathbf{5 - 2}}=\frac{\mathbf{6}}{\mathbf{3}}=\mathbf{2}$. Write your answer in slope intercept form.
Now let's use the point-slope form $\boldsymbol{y}-\boldsymbol{y}_{1}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{1}}\right)$. Which point do we use? Will it matter?
With $\boldsymbol{A}(\mathbf{2},-\mathbf{3})$ we get

$$
\begin{aligned}
y-(-3) & =2(x-2) \\
y+3 & =2 x-4 \\
y & =2 x-4-3 \\
y & =3 x-7
\end{aligned}
$$

With $\boldsymbol{B}(\mathbf{5}, \mathbf{3})$ we get

$$
\begin{aligned}
y-3 & =2(x-5) \\
y-3 & =2 x-10 \\
y & =3 x-10+3 \\
y & =3 x-7
\end{aligned}
$$

Example 3:
Find an equation of line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{A}(\mathbf{3}, \mathbf{4})$ and parallel to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{2 x} \boldsymbol{\mathbf { 5 }} \boldsymbol{y}=\mathbf{1 0}$. Write your answer in slope-intercept form.

## Solution:

Solve $\boldsymbol{L}_{\mathbf{1}}$ for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ to find the slope.

$$
\begin{aligned}
2 x-5 y & =10 \\
-5 y & =-2 x+10 \\
y & =\frac{-2}{-5} x-2 \\
y & =\frac{2}{5} x-2 \\
y & =m x+b
\end{aligned}
$$

The slope of the given line $\boldsymbol{L}_{\mathbf{1}}$ is also the slope of the desired line $\boldsymbol{L}_{\mathbf{2}}$, which passes through $\boldsymbol{A}(\mathbf{3}, \mathbf{4})$.
By the point-slope form

$$
\begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-(-4) & =\frac{2}{5}(x-3) \\
y+4 & =\frac{2}{5} x-\frac{2}{5}(3) \\
y & =\frac{2}{5} x-\frac{6}{5}-4 \\
y & =\frac{2}{5} x-\frac{6}{5}-\frac{20}{5} \\
y & =\frac{2}{5} x-\frac{26}{5}
\end{aligned}
$$

Note: If two lines $\boldsymbol{A}_{\mathbf{1}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{1}} \boldsymbol{y}=\boldsymbol{C}_{\mathbf{1}}$ and $\boldsymbol{A}_{\mathbf{2}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{2}} \boldsymbol{y}=\boldsymbol{C}_{\mathbf{2}}$ are parallel then $\boldsymbol{A}_{\mathbf{1}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{1}} \boldsymbol{y}=\boldsymbol{A}_{\mathbf{2}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{2}} \boldsymbol{y}$ because the slopes are $-\frac{\boldsymbol{A}_{\mathbf{1}}}{\boldsymbol{B}_{\mathbf{1}}}=-\frac{\boldsymbol{A}_{\mathbf{2}}}{\boldsymbol{B}_{\mathbf{2}}}$. The equations only differ by a constant $\boldsymbol{C}$ found with the help of the given point.
$L_{2}$ becomes $2 x-5 y=C$.
Use $\boldsymbol{A}(\mathbf{3}, \mathbf{4})$ to evaluate $\boldsymbol{C}$.

$$
\begin{aligned}
2 x-5 y & =C \\
2(3)-5(-4) & =C \\
6+20 & =C \\
C & =-26
\end{aligned}
$$

Write $\boldsymbol{L}_{\mathbf{2}}$ in slope-intercept form.

$$
\begin{aligned}
2 x-5 y & =-26 \\
2 x & =5 y-26 \\
2 x+26 & =5 y \\
5 y & =2 x-26 \\
y & =\frac{2}{5} x-\frac{26}{5}
\end{aligned}
$$

Example 4:
Find an equation of the line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{A}(\mathbf{3}, \mathbf{4})$ and perpendicular to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{2 x}-\mathbf{5} \boldsymbol{y}=\mathbf{1 0}$. Write your answer in slope-intercept form.

## Solution:

Solve $\boldsymbol{L}_{\mathbf{1}}$ for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ to find the slope.

$$
\begin{aligned}
2 x-5 y & =10 \\
-5 y & =-2 x+10 \\
y & =\frac{-2}{-5} x-2 \\
y & =\frac{2}{5} x-2 \\
y & =m x+b
\end{aligned}
$$

The slope of the desired line $\boldsymbol{L}_{2}$ is the negative reciprocal of $\boldsymbol{L}_{1}$. Since $\boldsymbol{m}_{1}=\frac{\mathbf{2}}{\mathbf{5}}, \boldsymbol{m}_{2}=-\frac{1}{\frac{2}{5}}=-\frac{\mathbf{5}}{\mathbf{2}}$.
$L_{2}$ passes through $\boldsymbol{A}(\mathbf{3},-4)$.
By the point-slope form

$$
\begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-(-4) & =-\frac{5}{2}(x-3) \\
y+4 & =-\frac{5}{2} x-\frac{5}{2}(-3) \\
y & =-\frac{5}{2} x+\frac{15}{2}-4 \\
y & =-\frac{5}{2} x+\frac{15}{2}-\frac{8}{2} \\
y & =-\frac{2}{5} x+\frac{7}{2}
\end{aligned}
$$

Example 5:
Rewrite $\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{y}=\boldsymbol{C}$ in slope-intercept form.

## Solution:

$$
\begin{aligned}
A x+B y & =C \\
B y & =-\boldsymbol{A} x+C \\
y & =-\frac{A}{B} x+\frac{C}{B}
\end{aligned}
$$

Example 6:
A train left the station at a speed of $\frac{\mathbf{3 0} \text { miles }}{\text { hour }}$.
At what time was the train $\mathbf{1 0 0}$ miles from the station?

## Solution:

Let's imagine a Cartesian coordinate system with time in hours horizontally on the $\boldsymbol{x}$-axis and distance in
miles vertically on the $\boldsymbol{y}$-axis. The train started at (0,0). The slope of the line is $\frac{\mathbf{3 0} \text { miles }}{\text { hour }}$. The equation of the distance $\boldsymbol{y}$ is $\boldsymbol{y}=\mathbf{3 0 \boldsymbol { x }}$

$$
\begin{aligned}
y & =30 x \\
100 & =30 x \\
30 x & =100 \\
x & =\frac{100}{30} \\
x & =3 \frac{1}{3} \text { hours }
\end{aligned}
$$

$x=3 \frac{1}{3}$ hours is equivalent to $\mathbf{3}$ hours and $\frac{\mathbf{6 0}}{\mathbf{3}}$ minutes or $\mathbf{3}$ hours and 20 minutes,
The train will be $\mathbf{1 0 0}$ miles from the station at $\mathbf{1 1}$ : $\mathbf{2 0}$ A.M.

### 25.5 Exercise 25

1. Find the equation of a line through $\boldsymbol{P}(\mathbf{0}, \mathbf{- 5})$ if the slope is $\boldsymbol{m}=\mathbf{7}$.
2. Find the equation of the line passing through $\boldsymbol{P}(\mathbf{1}, \mathbf{- 9})$ and $\boldsymbol{Q}(\mathbf{- 5}, \mathbf{3})$. Write your answer in slope intercept form.
3. Find an equation of the line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{P}(\mathbf{5}, \mathbf{- 6})$ and parallel to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{3 x}+\mathbf{4 y}=\mathbf{1 2}$. Write your answer in slope-intercept form.
4. Find an equation of the line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{P}(\mathbf{5}, \mathbf{- 6})$ and perpendicular to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{3 x}+\mathbf{4} \boldsymbol{y}=\mathbf{1 2}$.

Write your answer in slope-intercept form.
5. Rewrite $\boldsymbol{y}-\boldsymbol{y}_{\boldsymbol{1}}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{1}}\right)$ in general (or standard) form $\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{y}=\boldsymbol{C}$.
6. Rewrite $\boldsymbol{y}-\boldsymbol{y}_{\mathbf{1}}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\mathbf{1}}\right)$ in general (or standard) form.

A train left the station at $\mathbf{1 2}: \mathbf{0 0}$ A.M. and traveled at an average speed of $\mathbf{4 0}$ miles hour. At what time will the train be $\mathbf{2 0 0}$ miles from the station?

## STOP!

1. Find the equation of a line through $\boldsymbol{P}(\mathbf{0},-\mathbf{5})$ if the slope is $\boldsymbol{m}=\mathbf{7}$.

## Solution:

Note that $\boldsymbol{P}(\mathbf{0}, \mathbf{5})$ is the $\boldsymbol{y}$-intercept, so $\boldsymbol{b}=\mathbf{- 5}$ in the formula. The slope-intercept form $\boldsymbol{y}=\boldsymbol{m} \boldsymbol{x}+\boldsymbol{b}$ then becomes $\boldsymbol{y}=-\mathbf{7 x}-\mathbf{5}$.
Could you have used the point-slope form since you know one point and the slope?
$\boldsymbol{y}-\boldsymbol{y}_{1}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{1}\right)$ can be rewritten $\boldsymbol{y}-(-5)=-\mathbf{7}(\boldsymbol{x}-\mathbf{0})$ or $\boldsymbol{y}=-\boldsymbol{7} \boldsymbol{x}-\mathbf{5}$.
2. Find the equation of the line passing through $\boldsymbol{P}(\mathbf{1}, \mathbf{- 9})$ and $\boldsymbol{Q}(-\mathbf{5}, \mathbf{3})$. Write your answer in slope-intercept form.

## Solution:

It would be nice to know the slope.
Well, $m=\frac{3-(-9)}{-5-1}=\frac{12}{-6}=-2$.
Now let's use the point-slope form $\boldsymbol{y}-\boldsymbol{y}_{\boldsymbol{1}}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{1}}\right)$.
With $\boldsymbol{P}(\mathbf{1}, \mathbf{- 9})$ we get

$$
\begin{aligned}
y-(-9) & =-2(x-1) \\
y+9 & =-2 x+2 \\
y & =-2 x+2-9 \\
y & =-2 x-7
\end{aligned}
$$

With $\boldsymbol{Q}(-5,3)$ we get

$$
\begin{aligned}
y-3 & =-2[x-(-5)] \\
y-3 & =-2 x-10 \\
y & =-2 x-10+3 \\
y & =-2 x-7
\end{aligned}
$$

3. Find an equation of the line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{P}(\mathbf{5}, \mathbf{- 6})$ and parallel to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{3 x}+\mathbf{4 y}=\mathbf{1 2}$. Write your answer in slope-intercept form.

## Solution:

Solve $\boldsymbol{L}_{\mathbf{1}}$ for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ to find the slope.

$$
\begin{aligned}
3 x+4 y & =12 \\
4 y & =-3 x+12 \\
y & =\frac{-3}{4} x+\frac{12}{4} \\
y & =-\frac{3}{4} x+3 \\
y & =m x+b
\end{aligned}
$$

The slope $\boldsymbol{m}=-\frac{\mathbf{3}}{\mathbf{4}}$ of the given line $\boldsymbol{L}_{\mathbf{1}}$ is also the slope of the desired line $\boldsymbol{L}_{\mathbf{2}}$, which passes through $P(5,-6)$.

By the point-slope form

$$
\begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-(-6) & =-\frac{3}{4}(x-5) \\
y+6 & =-\frac{3}{4} x-\frac{3}{4}(-5) \\
y & =-\frac{3}{4} x+\frac{15}{4}-6 \\
y & =-\frac{3}{4} x+\frac{15}{4}-\frac{24}{4} \\
y & =-\frac{3}{4} x-\frac{9}{4}
\end{aligned}
$$

Note: If two lines $\boldsymbol{A}_{\mathbf{1}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{1}} \boldsymbol{y}=\boldsymbol{C}_{\mathbf{1}}$ and $\boldsymbol{A}_{\mathbf{2}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{2}} \boldsymbol{y}=\boldsymbol{C}_{\mathbf{2}}$ are parallel then
$\boldsymbol{A}_{1} \boldsymbol{x}+\boldsymbol{B}_{1} \boldsymbol{y}=\boldsymbol{A}_{\mathbf{2}} \boldsymbol{x}+\boldsymbol{B}_{\mathbf{2}} \boldsymbol{y}$. The equations only differ by a constant $\boldsymbol{C}$ found with the help of the given point $\boldsymbol{P}(\mathbf{5},-\mathbf{6})$.
$L_{2}$ becomes $\mathbf{3 x}+4 \boldsymbol{y}=\boldsymbol{C}$.
Use $\boldsymbol{P}(5,-6)$ to evaluate

$$
\begin{aligned}
3 x+4 y & =C \\
3(5)+4(-6) & =C \\
15-24 & =C \\
C & =-9
\end{aligned}
$$

Write $\boldsymbol{L}_{\mathbf{2}}$ in slope-intercept form.

$$
\begin{aligned}
3 x+4 y & =-9 \\
3 x & =-4 y-9 \\
3 x+9 & =-4 y \\
-4 y & =3 x+9 \\
y & =\frac{3}{-4} x+\frac{9}{-4} \\
y & =-\frac{3}{4} x-\frac{9}{4}
\end{aligned}
$$

4. Find an equation of the line $\boldsymbol{L}_{\mathbf{2}}$ containing $\boldsymbol{P}(\mathbf{5}, \mathbf{- 6})$ and perpendicular to $\boldsymbol{L}_{\mathbf{1}}: \mathbf{3 x}+\mathbf{4} \boldsymbol{y}=\mathbf{1 2}$.

Write your answer in slope-intercept form.

## Solution:

Solve $\boldsymbol{L}_{\mathbf{1}}$ for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ to find the slope.

$$
\begin{aligned}
3 x+4 y & =12 \\
4 y & =-3 x+12 \\
y & =\frac{-3}{4} x+\frac{12}{4} \\
y & =-\frac{3}{4} x+3 \\
y & =m x+b
\end{aligned}
$$

The slope of the desired line $\boldsymbol{L}_{\mathbf{2}}$ is the negative reciprocal of $\boldsymbol{L}_{\mathbf{1}}$. Since $\boldsymbol{m}_{\mathbf{1}}=-\frac{\mathbf{3}}{\mathbf{4}}$,
$m_{2}=-\frac{1}{-\frac{3}{4}}=\frac{4}{3}$.
$\boldsymbol{L}_{\mathbf{2}}$ passes through $\boldsymbol{P}(\mathbf{5}, \mathbf{- 6})$.
By the point-slope form

$$
\begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-(-6) & =\frac{4}{3}(x-5) \\
y+6 & =\frac{4}{3} x+\frac{4}{3}(-5) \\
y & =\frac{4}{3} x-\frac{20}{3}-6 \\
y & =\frac{4}{3} x-\frac{20}{3}-\frac{18}{3} \\
y & =\frac{4}{3} x-\frac{38}{3}
\end{aligned}
$$

5. Rewrite $\boldsymbol{y}-\boldsymbol{y}_{\boldsymbol{1}}=\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\boldsymbol{1}}\right)$ in general (or standard) form $\boldsymbol{A} \boldsymbol{x}+\boldsymbol{B} \boldsymbol{y}=\boldsymbol{C}$.

## Solution:

$$
\begin{aligned}
\boldsymbol{y}-\boldsymbol{y}_{1} & =\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{1}\right) & & \\
\boldsymbol{y} & =y_{1}+\boldsymbol{m}\left(\boldsymbol{x}-\boldsymbol{x}_{\mathbf{1}}\right) & & \text { Add } \boldsymbol{y}_{1} \text { to both sides } \\
\boldsymbol{y} & =\boldsymbol{y}_{1}+\boldsymbol{m} \boldsymbol{x}-\boldsymbol{m} \boldsymbol{x}_{\mathbf{1}} & & \text { Distribute } \boldsymbol{m} \\
\boldsymbol{y}-\boldsymbol{m} \boldsymbol{x} & =\boldsymbol{y}_{1}-\boldsymbol{m} \boldsymbol{x}_{\mathbf{1}} & & \text { Subtract } \boldsymbol{m} \boldsymbol{x} \text { from both sides } \\
\boldsymbol{m} \boldsymbol{x}+(-\mathbf{1}) \boldsymbol{y} & =\boldsymbol{m} \boldsymbol{x}_{\mathbf{1}}-\boldsymbol{y}_{\mathbf{1}} & & \text { Multiply both sides by } \mathbf{- 1} \\
& & & \\
\boldsymbol{C}=\boldsymbol{m} \boldsymbol{x}_{\mathbf{1}}-\boldsymbol{y}_{1} & & &
\end{aligned} \quad \begin{aligned}
& \text { Let } \boldsymbol{B}=-\mathbf{1}, \\
&
\end{aligned}
$$

6. A train left the station at 12:00 A.M. and traveled at an average speed of $\mathbf{4 0}$ miles hour. At what time will the train be $\mathbf{2 0 0}$ miles from the station?

## Solution:

Let's imagine a Cartesian coordinate system with time in hours horizontally on the $\boldsymbol{x}$-axis and distance in miles vertically on the $\boldsymbol{y}$-axis. The train starts at ( 0,0 ), The slope of the line is $\frac{\mathbf{4 0} \text { miles }}{\text { hour }}$. The equation of the distance $\boldsymbol{y}$ is $\boldsymbol{y}=\mathbf{4 0 x}$

$$
\begin{aligned}
y & =40 x \\
200 & =40 x \\
40 x & =200 \\
x & =\frac{200}{40}=x=5 \text { hours }
\end{aligned}
$$

The train will be $\mathbf{2 0 0}$ miles from the station at $\mathbf{8}+\mathbf{5}=\mathbf{1 3}=\mathbf{1 2}+\mathbf{1}$ or $\mathbf{1}$ P.M.

## Chapter 26

# Graphing Linear Inequalities in Two Variables 

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### 26.1 YouTube

https://www.youtube.com/playlist?list=PLC8A4C95E403019E6\&feature=view_all

### 26.2 Basics

We are interested in finding the region of all the points on a rectangular coordinate system whose coordinates satisfy a given inequality. We shade such a region.

Consider all the points in the $\boldsymbol{x} \boldsymbol{y}$ plane. A line (or really any closed non-overlapping curve) subdivides the plane into two regions. Let's refer to this line as a boundary (like the geographical boundary between the United States and Canada or Mexico).

All the points on the plane are represented by an ordered pair of coordinates $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y})$. Some of these points are above the boundary, some lie on the line, others are located below the boundary. The boundary is represented by a linear equation. The equation is identical to the inequality except that the inequality sign is replaced by an equal sign.

Draw the boundary on a Cartesian coordinate system by finding two points on the boundary-line (preferably the intercepts). Then pick a test point which does not lie on the boundary. The origin with $\boldsymbol{x}=\mathbf{0}$ and $\boldsymbol{y}=\mathbf{0}$ is the simplest test point. (If the boundary contains the origin, pick a test point on one of the axes. At least $\boldsymbol{x}=\mathbf{0}$ or $\boldsymbol{y}=\mathbf{0}$.)

Substitute the coordinates of the test point into the inequality. If the inequality is true, the test point
identifies the region to shade. All the points on the same side of the boundary as the test point have coordinates that satisfy the inequality.

If the test point does not satisfy the inequality, then shade the region on the other side of the boundary.

### 26.3 Examples

Example 1:
Shade the solution of $\mathbf{2 x}+\mathbf{3 y}>6$.

## Solution:

Use $\mathbf{2 x}+\mathbf{3 y}=\mathbf{6}$ to find the boundary. (You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.)

If $\boldsymbol{x}=\mathbf{0}$ then $\mathbf{3 y}=\mathbf{6}$ or $\boldsymbol{y}=\mathbf{2} . \boldsymbol{A}(\mathbf{0}, \mathbf{2})$ is the $\boldsymbol{y}$-intercept.

If $\boldsymbol{y}=\mathbf{0}$ then $\mathbf{2 x}=\mathbf{6}$ or $\boldsymbol{x}=\mathbf{3} . \boldsymbol{B}(\mathbf{3}, \mathbf{0})$ is the $\boldsymbol{x}$ intercept.


The boundary is dashed to indicate that the points on the line fail to satisfy the inequality. Using test point $\boldsymbol{T}(\mathbf{0}, \mathbf{0})$ leads to $\mathbf{2 ( 0 ) + 3 ( 0 ) > 6}$ or $\mathbf{0}>\mathbf{6}$ which is false. Since the test point is below the boundary we shade the region above the line. Also note that the boundary is dashed since the equal sign is not part of the inequality.

Example 2:
Shade the solution of
$2 x+3 y \leq 6$.

## Solution:

Use $2 \boldsymbol{x}+\mathbf{3 y}=\mathbf{6}$ to find the boundary. (You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.)

If $\boldsymbol{x}=\mathbf{0}$ then $\mathbf{3 y}=\mathbf{6}$ or $\boldsymbol{y}=\mathbf{2}$. Thus $\boldsymbol{A}(\mathbf{0}, \mathbf{2})$ is the $\boldsymbol{y}$-intercept.

If $\boldsymbol{y}=\mathbf{0}$ then $\mathbf{2 x}=\mathbf{6}$ or $\boldsymbol{x}=\mathbf{3}$. Thus $\boldsymbol{B}(\mathbf{3}, \mathbf{0})$ is the $\boldsymbol{x}$-intercept.


The boundary is solid to indicate that the points on the line satisfy the inequality.
 the boundary we shade the region below the line.

Example 3:
A scout troupe was awarded $\$ 420$ for admission tickets to Magic Mountain. A regular adult ticket costs $\$ 60$, while the admission for a child is $\$ \mathbf{3 5}$.

Set up an inequality showing the number of adults/children that can join for the trip. Stay within budget.
Then list the maximum number of adults/children that can go on the outing as pairs of adults and children.

## Solution:

Preamble:
Let $\boldsymbol{x}$ be the number of adults and $\boldsymbol{y}$ the number of children that go on the trip.

| Number of | Number of | Admission <br> of one | Total price <br> (by age group) |
| :---: | :---: | :---: | :---: |
| adults | $\boldsymbol{x}$ | $\mathbf{6 0}$ | $\mathbf{6 0} \boldsymbol{x}$ |
| children | $\boldsymbol{y}$ | $\mathbf{3 5}$ | $\mathbf{3 5} \boldsymbol{y}$ |

Equation (actually inequality):

Total admission $\leq$ budget.
$60 x+35 y \leq 420$

For the boundary with $\boldsymbol{x}=\mathbf{0}, \mathbf{6 0}(\mathbf{0})+\mathbf{3 5 y}=\mathbf{4 2 0}, \quad \mathbf{3 5 y}=\mathbf{4 2 0} \quad[1 \mathrm{pt}]$ or $\quad y=\frac{420}{35}=12 . A(0,12)$ is the $y$-intercept.

With $y=0,60 x+35(0)=420, \quad 60 x=420$
or $\quad \boldsymbol{x}=\frac{\mathbf{4 2 0}}{\mathbf{6 0}}=\mathbf{7} . \boldsymbol{B}(\mathbf{7}, 0)$ is the $\boldsymbol{x}$-intercept.

| Number of <br> adults | Max. number <br> of children | price <br> (adults) | price <br> (children) | total |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbf{0}$ | $\mathbf{1 2}$ |  | $\mathbf{1 2 \cdot 3 5}=\mathbf{4 2 0}$ | $\mathbf{4 2 0}$ |
| $\mathbf{1}$ | $\mathbf{1 0}$ | $\mathbf{1 \cdot 6 0}=\mathbf{6 0}$ | $\mathbf{1 0} \cdot \mathbf{3 5}=\mathbf{3 5 0}$ | $\mathbf{4 1 0}$ |
| $\mathbf{2}$ | $\mathbf{8}$ | $\mathbf{2 \cdot 6 0}=\mathbf{1 2 0}$ | $\mathbf{8 \cdot 3 5}=\mathbf{2 8 0}$ | $\mathbf{4 0 0}$ |
| $\mathbf{3}$ | $\mathbf{6}$ | $\mathbf{3 \cdot 6 0}=\mathbf{1 8 0}$ | $\mathbf{6 \cdot 3 5}=\mathbf{2 1 0}$ | $\mathbf{3 9 0}$ |
| $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{4 \cdot 6 0}=\mathbf{2 4 0}$ | $\mathbf{5 \cdot 3 5}=\mathbf{1 7 5}$ | $\mathbf{4 1 5}$ |
| $\mathbf{5}$ | $\mathbf{3}$ | $\mathbf{5 \cdot 6 0}=\mathbf{3 0 0}$ | $\mathbf{3 \cdot 3 5}=\mathbf{1 0 5}$ | $\mathbf{4 0 5}$ |
| $\mathbf{6}$ | $\mathbf{1}$ | $\mathbf{6 \cdot 6 0}=\mathbf{3 6 0}$ | $\mathbf{1 \cdot 3 5}=\mathbf{3 5}$ | $\mathbf{3 9 5}$ |
| $\mathbf{7}$ | $\mathbf{1}$ | $\mathbf{6 \cdot 6 0}=\mathbf{4 2 0}$ | $\mathbf{0 \cdot 3 5 = 0}$ | $\mathbf{4 2 0}$ |



### 26.4 Exercise 26

1. Shade the solution of $\mathbf{3 x}-4 y>12$.
2. Shade the solution of $\mathbf{3 x}-4 y \leq \mathbf{1 2}$.
3. A storage bin has a rectangular floor measuring $\mathbf{3 0} \mathrm{ft}$ by $\mathbf{2 0} \mathrm{ft}$. The bin is to be filled with two types of boxes:
A small box occupies $\mathbf{3 0} \mathrm{ft}^{2}$ of floor space, a large box needs $\mathbf{5 0} \mathrm{ft}^{2}$ of floor space.
What is the maximum number of large boxes that can be stored?
Set up an inequality reflecting the number of large and small boxes that can be stored.
Then make a list with numbers from this maximum to $\mathbf{0}$. What is the highest number of small boxes that correspond to each number of large boxes in your list if as little floor space as possible is wasted?

## STOP!

1. Shade the solution of $3 x-4 y>12$.

## Solution:

Use $3 x-4 y=12$ to find the boundary. (You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.)

If $x=0$ then $-4 y=12$ or $\boldsymbol{y}=-3$. Thus $\boldsymbol{A}(0,-3)$ is the $\boldsymbol{y}$-intercept.

If $\boldsymbol{y}=0$ then $3 x=12$ or $\boldsymbol{x}=4$. Thus $B(4,0)$ is the $\boldsymbol{x}$-intercept.


The boundary is dashed to indicate that the points on the line fail to satisfy the inequality.
 test point is above the boundary we shade the region below the line.
2. Shade the solution of $\mathbf{3 x}-\mathbf{4 y} \leq \mathbf{1 2}$.

## Solution:

Use $3 x-4 y=12$ to find the boundary. (You need not solve for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$.)

If $\boldsymbol{x}=0$ then $-4 y=12$ or $\boldsymbol{y}=-3$. Thus $\boldsymbol{A}(0,-3)$ is the $\boldsymbol{y}$-intercept.

If $y=0$ then $3 x=12$ or $x=4$. Thus $B(4,0)$ is the $\boldsymbol{x}$-intercept.

The boundary is solid to indicate that the points on the line satisfy the inequality.


Using test point $\boldsymbol{T}(\mathbf{0}, \mathbf{0})$ leads to $\mathbf{3 ( 0 )}-\mathbf{4 ( 0 )} \leq \mathbf{1 2}$ or $\mathbf{0} \leq \mathbf{1 2}$ which is true. Since the test point is below the boundary we shade the region below the line.
3. A storage bin has a rectangular floor measuring $\mathbf{3 0} \mathrm{ft}$ by $\mathbf{2 0} \mathrm{ft}$. The bin is to be filled with two types of boxes:
A small box occupies $\mathbf{3 0} \mathrm{ft}^{2}$ of floor space, a large box needs $50 \mathrm{ft}^{2}$ of floor space.
What is the maximum number of large boxes that can be stored?
Set up an inequality reflecting the number of large and small boxes that can be stored.
Then make a list with numbers from this maximum to $\mathbf{0}$. What is the highest number of small boxes that correspond to each number of large boxes in your list if as little floor space as possible is wasted?

## Solution:

Preamble:
The floor space of the rectangular storage bin is
$\mathbf{3 0} \cdot \mathbf{2 0}=\mathbf{3 0 0} \mathrm{ft}^{2}$. The maximum number of large boxes that fit is $\frac{\mathbf{6 0 0}}{\mathbf{5 0}}=\mathbf{1 2}$.
Let $\boldsymbol{x}$ be the number of large boxes and $\boldsymbol{y}$ the number of small boxes that fit in the storage bin. The maximum number for $\boldsymbol{x}$ is 12 .

| Number of | Number of | Floor space <br> per box | Total space <br> in ft $^{2}$ |
| :---: | :---: | :---: | :---: |
| Large boxes | $\boldsymbol{x}$ | $\mathbf{5 0}$ | $\mathbf{5 0} \boldsymbol{x}$ |
| Small boxes | $\boldsymbol{y}$ | $\mathbf{3 0}$ | $\mathbf{3 0} \boldsymbol{y}$ |

Equation (actually inequality):
Occupied space $\leq$ storage bin space.
$50 x+30 y \leq 600$
For the boundary with $x=0,50(0)+30 y=600, \quad 30 y=600$ or $\quad y=\frac{600}{30}=20 . A(0,20)$ is the $\boldsymbol{y}$-intercept.

With $y=0,50 x+30(0)=600$,
$\mathbf{5 0 x}=\mathbf{6 0 0}$ or
$x=\frac{\mathbf{6 0 0}}{\mathbf{5 0}}=\mathbf{1 2} \boldsymbol{B}(\mathbf{1 2}, 0)$ is the $\boldsymbol{x}$-intercept.

| Number of large boxes | Max. number of small boxes | Floor space (Large boxes) | Floor space (small boxes) | total space |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 20 | $\mathbf{0} \cdot 50=0$ | $20 \cdot 30=600$ | 600 |
| 1 | 18 | $1 \cdot 50=50$ | $18 \cdot 30=540$ | 600 |
| 2 | 16 | $2 \cdot 50=100$ | $16 \cdot 30=480$ | 580 |
| 3 | 15 | $\mathbf{3 \cdot 5 0}=150$ | $15 \cdot 30=450$ | 600 |
| 4 | 13 | $4 \cdot 50=200$ | $13 \cdot 30=390$ | 590 |
| 5 | 11 | $5 \cdot 50=250$ | $11 \cdot 30=330$ | 580 |
| 6 | 10 | $6 \cdot 50=300$ | $10 \cdot 30=300$ | 600 |
| 7 | 8 | $7 \cdot 50=350$ | $8 \cdot 30=240$ | 590 |
| 8 | 6 | $8 \cdot 50=400$ | $6 \cdot 30=180$ | 580 |
| 9 | 5 | $9 \cdot 50=450$ | $5 \cdot 30=150$ | 600 |
| 10 | 3 | $10 \cdot 50=500$ | $3 \cdot 30=90$ | 590 |
| 11 | 1 | $11 \cdot 50=550$ | $1 \cdot 30=30$ | 580 |
| 12 | 0 | $12 \cdot 50=600$ | $\mathbf{0} \cdot \mathbf{3 0}=0$ | 600 |

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## Chapter 27

## Introduction to Functions

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### 27.1 YouTube

https://www.youtube.com/playlist?list=PL1BA1D77D6AC8D457\&feature=view_all

### 27.2 Function Basics

Suppose you get paid $\$ \mathbf{1 0}$ for every hour you work.
Work 1 hour, get $\$ 10$.
Work 2 hours, get $2(\$ 10)$.
Work 3 hours, get $3(\$ 10)$.
$\ldots$ Work $\boldsymbol{t}$ hours, get $\boldsymbol{t}(\$ 10)=\$ \mathbf{1 0 t}$.
The second column depends uniquely on the first. You pick a natural number $(1,2,3, \cdots)$ and I'll tell you how much you earn. (From a practical point of view there is a limit on the number of hours you can work and you may get paid for working a fraction of an hour. We'll get to that.)

The "rule" that assigns pay to hours is an example of a function.
This rule is called a function. It is a mapping from the number of hours to pay.
The set of all hours $\boldsymbol{t}$ is called the domain of the function.
The set of corresponding value $\mathbf{1 0 t}$ is called the range.
The rule could be represented symbolically by $\boldsymbol{P}(\boldsymbol{t})=\mathbf{1 0 t}$.


Note: The symbol $\boldsymbol{P}(\boldsymbol{t})$ is read " $\boldsymbol{P}$ of $\boldsymbol{t}$ ", not $\boldsymbol{P}$ times
Of is normally translated to mean multiplication. Not here. "of" is ambiguous and depends on the context.
The set of parentheses ( ) is normally translated to mean multiplication. Not here. "( )" is ambiguous and depends on the context.
( ) Takes the argument of the function


The letter $\boldsymbol{P}$ is arbitrary. It was chosen here to remind us of Pay. The letter $\boldsymbol{t}$ is arbitrary. It was chosen here to denote time.
$\boldsymbol{f}(\boldsymbol{x})$ - read $\boldsymbol{f}$ of $\boldsymbol{x}$-is the generic symbol for a function. Functions are used throughout scientific literature. You will not appreciate its significance here, but learn the concept. Pick up any mathematical or scientific article and count the number of times a function is used.

A function is a unique association from an element of the domain to an element of the range. (We'll see the vertical line test shortly. We'll also introduce relations.)

So much for mapping.
Let's revisit the concept of a function $\boldsymbol{f}(\boldsymbol{x})$ by thinking of a machine.
Our machine $\boldsymbol{f}$ takes argument $\boldsymbol{x}$ as input. $\boldsymbol{x}$ is also called the independent variable.
The machine manipulates the argument according to the way it was designed by the rule of association (the argument $\boldsymbol{t}$ is multiplied by 10 in the Pay function above).

The output from the machine is $\boldsymbol{f}(\boldsymbol{x})$. The output is also called the dependent variable. The output from the machine depends on what we put into the machine.

If $f(x)=10 x$ then $f(1)=10, f(2)=10(2)=20$,
$f(3)=10(3), \ldots$
When you see $\boldsymbol{f}(\boldsymbol{x})=\mathbf{2 x}$ you may think of $\boldsymbol{f}(\boldsymbol{x})$ as $\boldsymbol{y}$.
$y=f(x)=2 x$.
$f(5)=50$ means $\boldsymbol{y}=50$ if $\boldsymbol{x}=5$.
Suppose you have two functions in the same problem.
The cost of manufacturing an item depends on number of employees, cost of material, utilities, ....
The profit from selling the item depends on the number of items sold, time of year, the taxes, the weather
(more umbrellas are sold during the rainy season), ....

### 27.3 Examples

Example 1:
Let the cost be $\boldsymbol{C}(\boldsymbol{x})=\mathbf{3 x}+\mathbf{4}$ and the profit is $\boldsymbol{P}(\boldsymbol{x})=\mathbf{5 x}+\mathbf{6}$. We want to find the cost and the profit if $\boldsymbol{x}=\mathbf{6}$.

## Solution:

(a) If we did not have the concept of functions:

We need to say that if $\boldsymbol{y}$ is cost and $\boldsymbol{x}=\mathbf{6}$ then $\boldsymbol{y}=\mathbf{3 x}+\mathbf{4}$ or $\boldsymbol{y}=\mathbf{3}(6)+4=18+4=\mathbf{2 2}$
and
if $\boldsymbol{y}$ is profit and $\boldsymbol{x}=\mathbf{6}$ then $\boldsymbol{y}=5 \boldsymbol{x}+6, \boldsymbol{y}=5(6)+6=30+6=36$
(b) With he concept of functions, the following is more compact:

Let the cost be $\boldsymbol{C}(\boldsymbol{x})=\mathbf{3 x}+\mathbf{4}$ and the profit is $\boldsymbol{P}(\boldsymbol{x})=\mathbf{5 x}+\mathbf{6}$. We want to find the cost and the profit if $\boldsymbol{x}=\mathbf{6}$.
$C(6)=3(6)+4=18+4=22$ and
$P(6)=5(6)+6=30+6=36$
Example 2:
If $y=f(x)=x^{2}+2 x-3$ find (a) $f(1)$, (b) $f(0)$, (c) $f(-1)$, (d) $f(t)$, (e) $f(t+1)$,

## Solution:


(a) $f(1)=(1)^{2}+2(1)-3=1+2-3=0$. We say $f(1)=0$, we mean $y=0$ if $x=1$.

(b) $f(0)=(0)^{2}+2(0)-3=0+0-3=-3$. We say $f(0)=-\mathbf{3}$, we mean $\boldsymbol{y}=-\mathbf{3}$ if $\boldsymbol{x}=\mathbf{0}$.

(c) $f(-1)=(-1)^{2}+2(-1)-3=1-2-3=-4$. We say $f(-1)=-4$, we mean $y=-4$ if $x=-1$.

(d) $f(t)=(t)^{2}+2(t)-\mathbf{3}=t^{2}+2 t-\mathbf{3}=0$. We say $f(t)=t^{2}+2 t-3$, we mean $y=t^{2}+2 t+\mathbf{3}$ if $\boldsymbol{x}=\boldsymbol{t}$.

(e) $f(t+1)=(t+1)^{2}+2(t+1)-3 \quad$ Develop $(t+1)^{2}$ in future lesson.
$=(t+1)(t+1)+2(t+1)-3 \quad$ Rewrite exponent as multiplication $=(t)(t+1)+(1)(t+1)+2(t+1)-3$ Distribute $1^{\text {st }}(t+1)$ over $2^{\text {nd }}$. $=t^{2}+t+t+1+2 t+2-\mathbf{3} \quad$ Keep distributing.
$=\boldsymbol{t}^{2}+4 \boldsymbol{t} \quad$ Combine "like-terms".
We say $f(t+1)=t^{2}+4 t$, we mean $y=t^{2}+4 t$ if $x=t+1$.


### 27.4 Relations and Vertical Line Test

Case 1:
$f:\{(\mathbf{1}, \mathbf{2}),(\mathbf{2}, \mathbf{5}),(\mathbf{3}, \mathbf{- 1})\}$ is an example of a function (mapping concept).
$1 \rightarrow 2$ or $f(1)=2$
$2 \rightarrow 5$ or $f(2)=5$
$3 \rightarrow-1$ or $f(3)=-1$

$f$ is a function. Each element of the domain 1, 2, 3 corresponds to a unique element in the range $\mathbf{2 5}, \mathbf{- 1}$.
$\boldsymbol{f}$ passes the vertical line test.

Case 2:
$\boldsymbol{g}:\{(\mathbf{1}, \mathbf{2}),(\mathbf{2}, \mathbf{5}),(\mathbf{3}, \mathbf{2})\}$ is an example of a function (mapping concept).
$\mathbf{1} \rightarrow \mathbf{2}$ or $g(1)=2$

$2 \rightarrow 5$ or $g(2)=5$
$\mathbf{3} \rightarrow \mathbf{2}$ or $\boldsymbol{g}(\mathbf{3})=\mathbf{2}$
$\boldsymbol{g}$ is a function. Each element of the domain $\mathbf{1 , 2 , 3}$ corresponds to a unique element in the range $\mathbf{2} 5$.
Case 3:
$h:\{(\mathbf{1}, \mathbf{2}),(\mathbf{1}, \mathbf{5}),(\mathbf{3}, \mathbf{- 1})\}$ is an example of a relation (mapping concept).
$\mathbf{1} \rightarrow \mathbf{2}$ or $\boldsymbol{g}(\mathbf{1})=\mathbf{2}$
$1 \rightarrow 5$ or $g(1)=5$

$3 \rightarrow-1$ or $g(3)=-1$
$\boldsymbol{h}$ is a not function. Each element of the domain $\mathbf{1}$ does not correspond to a unique element in the range. We say that $\boldsymbol{h}$ is a relation. It does not pass the Vertical Line Test.

## Example 4:

Is the area of a circle a function of its radius or is it a relation?

## Solution:

The area of a circle depends uniquely on its radius. The formula is $\boldsymbol{A}(\boldsymbol{R})=\boldsymbol{\pi} \boldsymbol{R}^{2} . \boldsymbol{A}(\boldsymbol{R})$ is a function.
Example 5:
Find a formula (a function) that gives the correct result for dosage of a medicine in terms of body weight. The following table is recommended by the manufacturer:

| Weight (lb) | $\mathbf{1 0 0}$ | $\mathbf{1 1 0}$ | $\mathbf{1 2 0}$ | $\mathbf{1 3 0}$ | $\mathbf{1 4 0}$ | $\mathbf{1 5 0}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Dosage mL | $\mathbf{1 5 0}$ | $\mathbf{1 6 4}$ | $\mathbf{1 7 8}$ | $\mathbf{1 9 2}$ | $\mathbf{2 0 6}$ | $\mathbf{2 2 0}$ |

Assume the function is linear.

## Solution:

Let's use the slope-intercept form of the equation of a line and (any) two points from the table. Pick $\boldsymbol{x}=\boldsymbol{w e i g h t}$ and $\boldsymbol{y}=$ dosage .

Use $\boldsymbol{A}(\mathbf{1 0 0}, 150)$ and $\boldsymbol{B}(\mathbf{1 4 0}, 206)$.
For the point-slope form of the equation of a line

$$
\begin{aligned}
m=\frac{206-150}{140-100} & =\frac{56}{40}=1.4 \frac{\mathrm{~mL}}{\mathrm{lb}} \\
y-y_{1} & =m\left(x-x_{1}\right) \\
y-150 & =1.4(x-100) \\
y-150 & =1.4 x-140) \\
y-150+150 & =1.4 x-140+150) \\
y & =1.4 x+10
\end{aligned}
$$

The formula is $f(x)=1.4 x+10$.
Example 6:
Draw a circle. Do you have the graph of a function or a relation? Every point on the circle is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.

## Solution:

Does the graph of a circle pass the vertical line test? No, it does not. The graph is that of a relation.
Vertical lines that intersect the circle (other than vertical tangents) intersect the circle in two points.
Two different values of $\boldsymbol{y}$ correspond to the same $\boldsymbol{x}$.
Example 7:
Draw a horizontal line. Do you have the graph of a function or a relation? Every point on the line is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.

## Solution:

Does the graph of a horizontal line pass the vertical line test? Yes, it does. The graph is that of a function.
The same value of $\boldsymbol{y}$ correspond to different values of $\boldsymbol{x}$, but there are no two different values of $\boldsymbol{y}$ for the same $\boldsymbol{x}$.

Example 8:
Does every line represent a function?

## Solution:

Every non-vertical line is the graph of a function.

### 27.5 Exercise 27

1. If $g(x)=2 \frac{1}{x}-4 x+9$ find
(a) $\boldsymbol{g}(\mathbf{1}),(\mathrm{b}) \boldsymbol{g}(\mathbf{0})$, (c) $\boldsymbol{g}(-\mathbf{1})$, (d) $\boldsymbol{g}(\boldsymbol{t})$, (e) $\boldsymbol{g}(\boldsymbol{t}-\mathbf{1})$,
2. Is $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{c}, \boldsymbol{d}),(\boldsymbol{e}, \boldsymbol{f})\}$ an example of a function (mapping concept)?
3. Is $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{b}, \boldsymbol{c}),(\boldsymbol{d}, \boldsymbol{b})\}$ an example of a function (mapping concept)?
4. Is $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{a}, \boldsymbol{c}),(\boldsymbol{d}, \boldsymbol{e})\}$ an example of a relation (mapping concept)?
5. Is the volume of a sphere a function of its radius or is it a relation?
6. Find a formula (a function) that gives the correct result for dosage of a medicine in terms of body surface area. The following table is recommended by the manufacturer:

| Surface area $\left(\mathrm{m}^{\mathbf{2}}\right)$ | $\mathbf{2}$ | $\mathbf{2 . 5}$ | $\mathbf{3}$ | $\mathbf{3 . 5}$ | $\mathbf{4}$ | $\mathbf{4 . 5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Dosage cc | $\mathbf{1 0}$ | $\mathbf{1 4}$ | $\mathbf{1 8}$ | $\mathbf{2 2}$ | $\mathbf{2 6}$ | $\mathbf{3 0}$ |

Assume the function is linear.
7. Draw a semi-circle. Do you have the graph of a function or a relation? Every point on the circle is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.
8. Does the graph of $\boldsymbol{x}=\mathbf{2}$ depicts the graph of a function or a relation? Every point on the line is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.
9. A plumber charges $\$ 90$ to come to your property. The first half-hour of repair is free. Every half-hour of repair beyond the first costs $\$ 40$. The plumber arrived at 1:00 P.M. At what time did he leave?
Let $\boldsymbol{x}$ represent the number of half-hours beyond the first one that the plumber worked at the site. Let $\boldsymbol{C}(\boldsymbol{x})$ represent the total charges.
(a) Express $\boldsymbol{C}(\boldsymbol{x})$ as a function.
(b) If the total charges amount to $\$ 290$, when did the plumber finish the repairs?

## STOP!

1. If $g(x)=2 \frac{1}{x}-4 x+9$ find
(a) $\boldsymbol{g}(\mathbf{1})$, (b) $\boldsymbol{g}(\mathbf{0})$, (c) $\boldsymbol{g}(-1)$, (d) $\boldsymbol{g}(\boldsymbol{t})$, (e) $\boldsymbol{g}(\boldsymbol{t}-\mathbf{1})$,

Solution:
$\boldsymbol{g}$ (name of machine)

(a) $\boldsymbol{g}(1)=\frac{2}{1}-4(1)+9=2-4+9=7$. We say $\boldsymbol{g}(1)=7$, we mean $\boldsymbol{y}=\mathbf{7}$ if $\boldsymbol{x}=1$.

(b) $\boldsymbol{g}(0)=\mathbf{2} \frac{1}{(0)}-4(0)+\mathbf{9}$ which is undefined. We say $\boldsymbol{g}(0)$ is undefined, we mean $\boldsymbol{y}$ is undefined if $\boldsymbol{x}=\mathbf{0}$.

(c) $g(-1)=2 \frac{1}{-1}-4(-1)+9=-2+4+9=11$. We say $\boldsymbol{g}(-1)=11$, we mean $\boldsymbol{y}=11$ if $x=-1$.

(d) $g(t)=2 \frac{1}{(t)}-4(t)+\mathbf{9}=0$. We say $g(t)=2 \frac{1}{t}-4 t+\mathbf{9}$, we mean $\boldsymbol{y}=2 \frac{1}{t}-4 t+\mathbf{9}$ if $\boldsymbol{x}=\boldsymbol{t}$.


$$
\left(2 \frac{1}{t}-4 t+9\right)
$$

(e) We say $g(t+1)=2 \frac{1}{t+1}-4(t+1)+9=\frac{1}{t+1}-4 t-4+9=\frac{1}{t+1}-4 t+5$, we mean $y=\frac{1}{t+1}-4 t+5$ if $x=t+1$.


$$
\left(2 \frac{1}{t+1}-4 t+5\right)
$$

2. $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{c}, \boldsymbol{d}),(\boldsymbol{e}, \boldsymbol{f})\}$ is an example of a function (mapping concept).

## Solution:

$\boldsymbol{a} \rightarrow \boldsymbol{b}$ or $\boldsymbol{h}(\boldsymbol{a})=\boldsymbol{b}$
$\boldsymbol{c} \rightarrow \boldsymbol{d}$ or $\boldsymbol{h}(\boldsymbol{c})=\boldsymbol{d}$
$\boldsymbol{e} \rightarrow \boldsymbol{f}$ or $\boldsymbol{h}(\boldsymbol{e})=\boldsymbol{f}$
$\boldsymbol{h}$ is a function. Each element of the domain $\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{e}$ corresponds to a unique element in the range $\boldsymbol{b} \boldsymbol{d}, \boldsymbol{f}$.
$\boldsymbol{f}$ passes the vertical line test.

3. Is $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{b}, \boldsymbol{c}),(\boldsymbol{d}, \boldsymbol{b})\}$ an example of a function (mapping concept)?

Solution:
$\boldsymbol{a} \rightarrow \boldsymbol{b}$ or $\boldsymbol{h}(\boldsymbol{a})=\boldsymbol{b}$
$\boldsymbol{b} \rightarrow \boldsymbol{c}$ or $\boldsymbol{h}(\boldsymbol{b})=\boldsymbol{c}$
$\boldsymbol{d} \rightarrow \boldsymbol{b}$ or $\boldsymbol{h}(\boldsymbol{d})=\boldsymbol{b}$

$\boldsymbol{g}$ is a function. Each element of the domain $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}$ corresponds to a unique element in the range $\boldsymbol{b} \boldsymbol{c}$.
4. Is $\boldsymbol{h}:\{(\boldsymbol{a}, \boldsymbol{b}),(\boldsymbol{a}, \boldsymbol{c}),(\boldsymbol{d}, \boldsymbol{e})\}$ an example of a relation (mapping concept)?

## Solution:

$\boldsymbol{a} \rightarrow \boldsymbol{b}$ or $\boldsymbol{h}(\boldsymbol{a})=\boldsymbol{b}$
$\boldsymbol{a} \rightarrow \boldsymbol{c}$ or $\boldsymbol{h}(\boldsymbol{a})=\boldsymbol{c}$
$\boldsymbol{d} \rightarrow \boldsymbol{e}$ or $\boldsymbol{h}(\boldsymbol{d})=\boldsymbol{e}$
$\boldsymbol{h}$ is a not function. Each element of the domain $\boldsymbol{a}$ does not correspond to a unique element in the range. We say that $\boldsymbol{h}$ is a relation.
 It does not pass the Vertical Line Test.
5. Is the volume of a sphere a function of its radius or is it a relation?

## Solution:

The volume of a sphere depends uniquely on its radius. The formula is $V(\boldsymbol{R})=\frac{\mathbf{4}}{\mathbf{3}} \boldsymbol{\pi} \boldsymbol{R}^{\mathbf{3}} . \boldsymbol{V}(\boldsymbol{R})$ is a function.
6. Find a formula (a function) that gives the correct result for dosage of a medicine in terms of body surface area. The following table is recommended by the manufacturer:

| Surface area $\left(\mathrm{m}^{2}\right)$ | $\mathbf{2}$ | $\mathbf{2 . 5}$ | $\mathbf{3}$ | $\mathbf{3 . 5}$ | $\mathbf{4}$ | $\mathbf{4 . 5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Dosage cc | $\mathbf{1 0}$ | $\mathbf{1 4}$ | $\mathbf{1 8}$ | $\mathbf{2 2}$ | $\mathbf{2 6}$ | $\mathbf{3 0}$ |

Assume the function is linear.

## Solution:

Let's use the slope-intercept form of the equation of a line and (any) two points from the table. Pick $\boldsymbol{x}=$ surface area and $\boldsymbol{y}=$ dosage.
Use $\boldsymbol{A}(\mathbf{2}, \mathbf{1 0})$ and $\boldsymbol{B}(4,26)$.
For the point-slope form of the equation of a line $\boldsymbol{m}=\frac{\mathbf{2 6 - 1 0}}{4-2}=\frac{16}{2}=8 \frac{\mathrm{cc}}{\mathrm{m}^{2}}$

$$
\begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
y-26 & =8(x-4) \\
y-26 & =8 x-32) \\
y-26+26 & =8 x-32+26 \\
y & =8 x-6
\end{aligned}
$$

The formula is $f(x)=8 x-8$.
7. Draw a semi-circle. Do you have the graph of a function or a relation? Every point on the circle is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.

## Solution:

The answer is yes, the graph of the semi-circle depicts a function only if a circle is cut by a horizontal line passing through its center.
If the semi-circle is formed by cutting the circle with a rising or a falling line, some vertical lines will intersect the semi-circle in two points.
Two different values of $\boldsymbol{y}$ correspond to the same $\boldsymbol{x}$.
8. Does the graph of $\boldsymbol{x}=\mathbf{2}$ depicts the graph of a function or a relation? Every point on the line is represented by an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$.

## Solution:

Does the graph of a vertical line pass the vertical line test? No, it does not. The graph is that of a relation.
The same value of $\boldsymbol{x}$ correspond to different values of $\boldsymbol{y}$, that is on the vertical line.
9. A plumber charges $\$ 90$ to come to your property. The first half-hour of repair is free. Every half-hour of repair beyond the first costs $\$ 40$. The plumber arrived at 1:00 P.M. At what time did he leave?
Let $\boldsymbol{x}$ represent the number of half-hours beyond the first one that the plumber worked at the site. Let $\boldsymbol{C}(\boldsymbol{x})$ represent the total charges.
(a) Express $\boldsymbol{C}(\boldsymbol{x})$ as a function.
(b) If the total charges amount to $\$ 290$, when did the plumber finish the repairs?

## Solution:

(a) $C(x)=40 x+90$
(b) $290=40 x+90$

$$
290-90=40 x+90-90
$$

$200=40 x$
$x=\frac{200}{40}=5$
The plumber spent a total of $\mathbf{5}+\mathbf{1}=\mathbf{6}$ half-hours or 3 hours. He left at $\mathbf{1}: \mathbf{0 0}+\mathbf{3}=\mathbf{4}: \mathbf{0 0}$ P.M.

## Chapter 28

## Laws/Properties of Exponents

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### 28.1 YouTube

https://www.youtube.com/playlist?list=PL387C744174D67C91\&feature=view_all

### 28.2 Logical Formulas

You should see from the numerical examples to follow that the rules can be generalized. The rules (or laws, or properties) will apply to multiplication of polynomials in this course. The rules are logical when the exponents are counting numbers $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots)$ but they are also useful with negative integer exponents and fractional or decimal exponents. We'll stick to integers for right now.

Case 1:
$\left(3^{2}\right)\left(3^{4}\right)=(\underbrace{(3 \cdot 3)}_{2} \underbrace{(3 \cdot 3 \cdot 3 \cdot 3)}_{4}=3^{2+4}=3^{6}$
In general

$$
\left(a^{m}\right) \cdot\left(a^{n}\right)=(\underbrace{a \cdot a \cdots a)}_{m} \underbrace{(a \cdot a \cdot a \cdots a)}_{n}=a^{m+n}
$$

Case 2:
$\left(3^{2}\right)^{4}=\left(3^{2}\right)\left(3^{2}\right)\left(3^{2}\right)\left(3^{2}\right)=\overbrace{(\underbrace{3 \cdot 3)}(\underbrace{3 \cdot 3)}(\underbrace{3 \cdot 3)}(\underbrace{3 \cdot 3)}}=3^{2 \cdot 4}=3^{8}$
In general
$2+2+2+2=2 \cdot 4=8$

$$
\begin{aligned}
\left(a^{m}\right)^{n}=\overbrace{(\underbrace{a \cdot a \cdots a)}_{m} \underbrace{(a \cdot a \cdots \cdots a)}_{m} \cdots \cdot(\underbrace{a \cdot a \cdots a)}_{m}}=a^{m \cdot n} \\
=m \cdot n
\end{aligned}
$$

Case 3:

$$
(3 \cdot 4)^{2}=(3 \cdot 4)(3 \cdot 4)=(3 \cdot 3)(4 \cdot 4)=3^{2} \cdot 4^{2}
$$

In general

$$
(a \cdot b)^{m}=\underbrace{(a \cdot b)(a \cdot b) \cdots(a \cdot b)}_{m}=\underbrace{(a \cdot a \cdot a \cdots a)}_{m} \underbrace{(b \cdot b \cdot b \cdots b)}_{m}=a^{m} \cdot b^{m}
$$

Note: $144=12^{2}=(3 \cdot 4)^{2}=3^{2} \cdot 4^{2}=9 \cdot 16=144$
But Warning

$\left(\frac{5}{7}\right)^{3}=\left(\frac{5}{7}\right) \cdot\left(\frac{5}{7}\right) \cdot\left(\frac{5}{7}\right)=\frac{5^{3}}{7^{3}}$
In general

$$
\left(\frac{a}{b}\right)^{m}=\left(\frac{a}{b}\right) \cdot\left(\frac{a}{b}\right) \cdots\left(\frac{a}{b}\right)=\frac{a^{m}}{b^{m}}
$$

Case 4 has interesting situations.

$$
\begin{aligned}
& \frac{2^{7}}{2^{4}}=\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2}| | \frac{a^{m}}{a^{n}}=\frac{a \cdot a \cdots a \cdots \cdots a}{a \cdot a \cdots a} \\
& \left.=\frac{\not 2 \cdot \not 2 \cdot \not 2 \cdot \not 2 \cdot 2 \cdot 2 \cdot 2}{\not 2 \cdot \not 2 \cdot \not 2 \cdot 22} \right\rvert\,=\frac{\not a \cdot \not a \cdots \not a \cdots \cdots a}{\not a \cdot \not a \cdots a} \\
& =2^{7-4} \\
& =2^{3} \\
& =a^{m-n} \\
& =\boldsymbol{a}^{\text {exp. of num. }}-\text { exp. of den. }
\end{aligned}
$$

Case 4a:

$$
\begin{aligned}
\frac{2^{4}}{2^{7}} & =\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} \\
& =\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{1}{2^{3}} \\
& =2^{4-7} \\
& =2^{-3}
\end{aligned}
$$

Steps 2 and 4 are equivalent, thus

$$
a^{-m}=\frac{1}{a^{m}}
$$

Case 4b:

$$
\begin{aligned}
\frac{2^{4}}{2^{4}} & =\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2} \\
& =\frac{22 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{1}{1} \\
& =2^{4-4} \\
2^{0} & =1
\end{aligned}
$$

Steps 2 and 4 are equivalent, thus

$$
\text { If } a \neq 0 \text { then } a^{0}=\mathbf{1}
$$

### 28.3 Examples

Example 1:
Simplify $\boldsymbol{a}^{\mathbf{3}} \cdot \boldsymbol{a}^{\mathbf{7}}$. Do not leave negative exponents.

## Solution:

$a^{3} \cdot a^{7}=a^{3+7}=a^{10}$.
Example 2:
Simplify $\left(a^{3}\right)^{7}$. Do not leave negative exponents.

## Solution:

$\left(a^{3}\right)^{7}=a^{3.7}=a^{21}$.
Example 3:
Simplify $\left(\boldsymbol{a}^{\mathbf{3}} \cdot \boldsymbol{a}^{\mathbf{7}}\right)^{\mathbf{5}}$. Do not leave negative exponents.
Solution:

$$
\begin{aligned}
\left(a^{3} \cdot a^{7}\right)^{5} & =\left(a^{3+7}\right)^{5} \\
& =\left(a^{10}\right)^{5} \\
& =a^{(10)(5)} \\
& =a^{50}
\end{aligned}
$$

Example 4:
Simplify $\left(\boldsymbol{a}^{\mathbf{3}} \cdot \boldsymbol{b}^{\mathbf{7}}\right)^{\mathbf{5}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(a^{3} \cdot b^{7}\right)^{5} & =\left(a^{3 \cdot 5}\right)\left(a^{7 \cdot 5}\right) \\
& =a^{15} b^{35}
\end{aligned}
$$

Example 5:
Simplify $\left(\frac{\boldsymbol{a}^{\mathbf{3}}}{\boldsymbol{a}^{\mathbf{7}}}\right)^{\mathbf{5}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{a^{3}}{a^{7}}\right)^{5} & =\left(a^{3-7}\right)^{5} \\
& =\left(a^{-4}\right)^{5} \\
& =a^{(-4)(5)} \\
& =a^{-20} \\
& =\frac{1}{a^{20}}
\end{aligned}
$$

Example 6:

Simplify $-\left(\frac{a^{3}}{a^{7}}\right)^{0}$. Do not leave negative exponents.

## Solution:

$$
-\left(\frac{a^{3}}{a^{7}}\right)^{0}=-1
$$

Example 7:
Simplify $\left(\frac{a^{3} b^{-4}}{a^{-7} b^{5}}\right)^{2}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{a^{3} b^{-4}}{a^{-7} b^{5}}\right)^{2} & =\left(a^{3-(-7)} b^{-4-5}\right)^{2} \\
& =\left(a^{10} b^{-9}\right)^{2} \\
& =\left(\frac{a^{10}}{b^{9}}\right)^{2} \\
& =\frac{a^{10 \cdot 2}}{b^{9 \cdot 2}} \\
& =\frac{a^{20}}{b^{18}}
\end{aligned}
$$

Here is a preview of uses of fractional exponents:
Define $\boldsymbol{a}^{1 / n}=\sqrt[n]{\boldsymbol{a}}$.
$49^{1 / 2}=\sqrt{49}=7$
$64^{1 / 2}=\sqrt[3]{64}=4$

Example 8:
Simplify $\mathbf{6 4}{ }^{\mathbf{1 / 2}} \cdot \mathbf{6 4} \mathbf{4}^{1 / \mathbf{3}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
64^{1 / 2} \cdot 64^{1 / 3} & =\sqrt{64} \cdot \sqrt[3]{64} \\
& =8 \cdot 4 \\
& =32
\end{aligned}
$$

Example 9:
The following is one story about the invention of Chess.

A young prince, a child about five years of age, became heir to a throne. He asked his mentor to invent a unique, clever, entertaining game. If the prince were satisfied with the game, he would grant the mentor any wish.

The mentor decided to teach the prince a lesson. He came up with the game of chess which involved a board with 64 squares. He asked the prince for a grain of wheat on the first square, double that (two grains) on the second square, double that (four grains on the third square), etc. till all $\mathbf{6 4}$ squares were filled. How many grains of wheat should be on the $\mathbf{6 4}{ }^{\text {th }}$ square? You may use a calculator.

## Solution:

| Square $\mathbf{1}$ | Square $\mathbf{2}$ | Square $\mathbf{3}$ | $\cdots$ | Square $\mathbf{6 4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbf{2}^{\mathbf{0}}$ | $\mathbf{2}^{\mathbf{1}}$ | $\mathbf{2}^{\mathbf{2}}$ | $\boldsymbol{\cdots}$ | $\mathbf{2}^{\mathbf{6 3}}$ |

$2^{\mathbf{6 3}}=\mathbf{9}, \mathbf{2 2 3}, \mathbf{3 7 2}, \mathbf{0 3 6}, \mathbf{8 5 4}, \mathbf{7 7 5}, 808 \approx \mathbf{9 . 2} \times \mathbf{1 0}^{\mathbf{1 8}}$ grains, which exceeded the amount of wheat available in the whole world (at that time.) The young prince could not keep his promise of granting the mentor's wish.

### 28.4 Exercise 28

1. Simplify $\boldsymbol{c}^{\mathbf{9}} \cdot \boldsymbol{c}^{\boldsymbol{- 7}}$. Do not leave negative exponents.
2. Simplify $\left(\boldsymbol{b}^{\mathbf{2}}\right)^{-\mathbf{3}}$. Do not leave negative exponents.
3. Simplify $\left(\boldsymbol{d}^{-4} \cdot \boldsymbol{d}^{\mathbf{7}}\right)^{\mathbf{- 1}}$. Do not leave negative exponents.
4. Simplify $\left(\boldsymbol{p}^{\mathbf{6}} \cdot \boldsymbol{q}^{-\mathbf{2}}\right)^{\mathbf{5}}$. Do not leave negative exponents.
5. Simplify $\left(\frac{\boldsymbol{a}^{\boldsymbol{c}}}{\boldsymbol{a}^{\boldsymbol{d}}}\right)^{\boldsymbol{p}} . \boldsymbol{c}>\boldsymbol{d}, \boldsymbol{p}>\mathbf{0}$. Do not leave negative exponents.
6. Simplify $\left(\frac{\boldsymbol{a}^{\boldsymbol{c}}}{\boldsymbol{a}^{\boldsymbol{d}}}\right)^{\boldsymbol{p}} . \boldsymbol{c}<\boldsymbol{d}, \boldsymbol{p}>\mathbf{0}$. Do not leave negative exponents.
7. Simplify $\left(-\frac{\boldsymbol{u}^{\mathbf{0}}}{\boldsymbol{v}^{-\mathbf{2}}}\right)^{\mathbf{4}}$. Do not leave negative exponents.
8. Simplify $\left(\frac{\boldsymbol{x}^{-\mathbf{7}} \boldsymbol{y}^{\mathbf{5}}}{\boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{-4}}\right)^{\mathbf{2}}$. Do not leave negative exponents.
9. Simplify $\mathbf{4}, \mathbf{0 9 6}^{\mathbf{1 / 2}} \cdot \mathbf{4}, \mathbf{0 9 6} \mathbf{1}^{\mathbf{1 / 3}}$. Do not leave negative exponents.

## STOP!

1. Simplify $\boldsymbol{c}^{\mathbf{9}} \cdot \boldsymbol{c}^{\boldsymbol{- 7}}$. Do not leave negative exponents.

## Solution:

$$
c^{9} \cdot c^{-7}=c^{9+(-7)}=c^{2}
$$

2. Simplify $\left(\boldsymbol{b}^{\mathbf{2}}\right)^{-\mathbf{3}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(b^{2}\right)^{-3} & =b^{2(-3)} \\
& =b^{-6} \\
& =\frac{1}{b^{6}}
\end{aligned}
$$

3. Simplify $\left(\boldsymbol{d}^{-\mathbf{4}} \cdot \boldsymbol{d}^{\mathbf{7}}\right)^{\mathbf{- 1}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(d^{-4} \cdot d^{7}\right)^{-1} & =\left(d^{-4+7}\right)^{-1} \\
& =\left(d^{3}\right)^{-1} \\
& =d^{-3} \\
& =\frac{1}{d^{3}}
\end{aligned}
$$

4. Simplify $\left(\boldsymbol{p}^{\mathbf{6}} \cdot \boldsymbol{q}^{-2}\right)^{\mathbf{5}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(p^{6} \cdot q^{-2}\right)^{5} & =p^{6 \cdot 5} \cdot q^{-2 \cdot 5} \\
& =p^{30} \cdot q^{-10} \\
& =\frac{p^{30}}{q^{10}}
\end{aligned}
$$

5. Simplify $\left(\frac{\boldsymbol{a}^{\boldsymbol{c}}}{\boldsymbol{a}^{\boldsymbol{d}}}\right)^{\boldsymbol{p}} . \boldsymbol{c}>\boldsymbol{d}, \boldsymbol{p}>\mathbf{0}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{a^{c}}{a^{d}}\right)^{p} & =\left(a^{c-d}\right)^{p} \\
& =a^{p(c-d)} \quad p>0 \text { and } c-d>0
\end{aligned}
$$

6. Simplify $\left(\frac{\boldsymbol{a}^{\boldsymbol{c}}}{\boldsymbol{a}^{\boldsymbol{d}}}\right)^{\boldsymbol{p}} . \boldsymbol{c}<\boldsymbol{d}, \boldsymbol{p}>\mathbf{0}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{a^{c}}{a^{d}}\right)^{p} & =\left(a^{c-d}\right)^{p} \\
& =a^{p(c-d)} \quad p>0 \text { and } c-d<0 \\
& =\frac{1}{a^{p(d-c)}} \quad p>0 \text { and } d-c>0
\end{aligned}
$$

7. Simplify $\left(-\frac{\boldsymbol{u}^{\mathbf{0}}}{\boldsymbol{v}^{-\mathbf{2}}}\right)^{\mathbf{4}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(-\frac{u^{0}}{v^{-2}}\right)^{4} & =\left(-\frac{1}{v^{-2}}\right)^{4} \\
& =\left(-\frac{v^{2}}{1}\right)^{4} \\
& =(-1)^{4} v^{2 \cdot 4} \\
& =v^{8}
\end{aligned}
$$

8. Simplify $\left(\frac{\boldsymbol{x}^{-7} \boldsymbol{y}^{\mathbf{5}}}{\boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{-4}}\right)^{\mathbf{2}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{x^{-7} y^{5}}{x^{3} y^{-4}}\right)^{2} & =\left(\frac{x^{-7-3} y^{5-(-4)}}{1}\right)^{2} \\
& =\left(\frac{x^{-10} y^{5+4}}{1}\right)^{2} \\
& =\left(\frac{y^{9}}{x^{10}}\right)^{2} \\
& =\frac{y^{9 \cdot 2}}{x^{10 \cdot 2}} \\
& =\frac{y^{18}}{x^{20}}
\end{aligned}
$$

9. Simplify $\mathbf{4}, \mathbf{0 9 6}^{\mathbf{1 / 2}} \cdot \mathbf{4}, \mathbf{0 9 6}^{\mathbf{1 / 3}}$. Do not leave negative exponents.

## Solution:

$$
\begin{aligned}
4,096^{1 / 2} \cdot 4,096^{1 / 3} & =\sqrt{4,096} \cdot \sqrt[3]{4,096} \\
& =64 \cdot 16 \\
& =1,024
\end{aligned}
$$

## Chapter 29

## Polynomials

(C) H. Feiner 2011

### 29.1 YouTube

https://www.youtube.com/playlist?list=PL4FD7D3D73F372E2F\&feature=view_all

### 29.2 Basics

A polynomial is a particular expression. It is a sum or difference of special terms. (Terms are quantities that are added to or subtracted from each other.) Each term is the product of a coefficient (a numerical factor) and one or more variables raised to a whole number.
$\mathbf{2} \boldsymbol{x}^{\mathbf{3}}$ is such a term. $\mathbf{2}$ is the coefficient and the exponent $\mathbf{3}$ is the whole number exponent. The exponent is called the degree.
$\mathbf{- 2} \boldsymbol{x}^{\mathbf{5}} \boldsymbol{y}^{\mathbf{4}}$ is another such term. The coefficient is $\mathbf{- 2}$ and the degree $\mathbf{9}$ (the sum of the exponents).
We shall concentrate on polynomials in one variable. The exponents are written in descending (or increasing) order of exponents.
$5 x^{7}-6 x^{4}+x^{3}+9 x-1$ is fine. So is $-1+9 x+x^{3}-6 x^{4}+5 x^{7}$, but not $5 x^{7}+9 x+x^{3}-6 x^{4}-1$.
The polynomial $5 x^{6}-4 x^{3}+7 x^{2}+8$ has four terms. The leading coefficient (the coefficient of the largest degree) is $\mathbf{5}$. The degree of the polynomial (the largest exponent) is $\mathbf{6}$.

The coefficient of the third degree term is $\mathbf{- 4}$ (this is also the coefficient of the second term.)
The coefficient of the second degree term is $\mathbf{7}$. The coefficient of the fifth degree term is $\mathbf{0}$.

The constant is $\mathbf{8}$.
The general form of a sixth degree polynomial is

$$
a_{6} x^{6}+a_{5} x^{5}+a_{4} x^{4}+a_{3} x^{3}+a_{2}^{2}+a_{1} x+a_{0}
$$

Matching our last expression above with
$5 x^{6}+0 x^{5}+0 x^{4}+(-4) x^{3}+7 x^{2}+0 x+8$ leads to
$a_{6}=5, a_{5}=0, a_{4}=0, a_{3}=-4, a_{2}=7, a_{1}=0, a_{0}=8$.
$\mathbf{6} \boldsymbol{x}^{\mathbf{4}}+\mathbf{5} \boldsymbol{x}^{-\mathbf{3}}-\mathbf{2 x}+\mathbf{9}$ is not a polynomial because the exponent $-\mathbf{3}$ is not a whole number.
$\mathbf{6} \boldsymbol{x}^{\mathbf{4}}+\mathbf{5}\left(\frac{\mathbf{4}}{\boldsymbol{x}^{\mathbf{3}}}\right)-\mathbf{2 x}+\mathbf{9}$ is not a polynomial because the exponent $-\mathbf{3}$ is not a whole number.
Recall that $\frac{\mathbf{1}}{\boldsymbol{x}^{\mathbf{3}}}=x^{-\mathbf{3}}$.
$6 \boldsymbol{x}^{4}+5^{-2} \boldsymbol{x}^{\mathbf{3}}-\frac{\mathbf{2}}{\mathbf{3}^{7}} \boldsymbol{x}+\mathbf{9}$ is a polynomial because the exponents are all whole numbers and there is no restriction on coefficients.

A polynomial with three terms is a trinomial. (A tricycle has three wheels).
A polynomial with two terms is a binomial. (A bicycle has two wheels).
A polynomial with one term is a monomial. (A monorail has one rail.)
$\mathbf{4 x}^{2} \boldsymbol{y}^{\mathbf{3}}-\mathbf{5} \boldsymbol{x} \boldsymbol{y}^{\mathbf{7}}-\mathbf{8 x}+\mathbf{9} \boldsymbol{y}^{\mathbf{1 0}}-\mathbf{1}$ is another example of a polynomial.
Polynomials are used in many mathematical an scientific situations. The speed ball your favorite baseball player throws follows a trajectory like $f(x)=9.8 x^{2}+49 x+5$.

Evaluating a polynomial means finding the value of the polynomial when the value of $\boldsymbol{x}$ is given.

### 29.3 Examples

Example 1:
Give an example of a fifth degree binomial with leading coefficient 3. (The answer is not unique.)

## Solution:

$\mathbf{3}^{\mathbf{5}}+\boldsymbol{a} \boldsymbol{x}^{\boldsymbol{m}}$ is such a polynomial. $\boldsymbol{a}$ can be any real number and $\boldsymbol{m}$ is a whole number $\neq \mathbf{5}$.
Example 2:
Evaluate the fourth degree polynomial $f(x)=-2 x^{4}+5 x^{2}-3$ if $x=-2$.

## Solution:

$$
\begin{aligned}
f(x) & =-2 x^{4}+5 x^{2}-3 \\
f(-2) & =-2(-2)^{4}+5(-2)^{2}-3 \\
& =-2(16)+5(4)-3 \\
& =-32+20-3 \\
& =-15
\end{aligned}
$$

Example 3:
A projectile is thrown upward with velocity $48 \frac{\mathrm{ft}}{\mathrm{sec}}$ from a platform 32 ft above ground. The equation of its path is
$H(t)=-16 t^{2}+48 t+32 \mathrm{ft}$.
How high above ground will the projectile be $\mathbf{3}$ seconds after launch?

## Solution:

$$
\begin{aligned}
H(t) & =-16 t^{2}+48 t+32 \\
H(3) & =-16(3)^{2}+48(3)+32 \\
& =-16(9)+144+32 \\
& =-144+144+32 \\
& =32
\end{aligned}
$$

The projectile will be $\mathbf{3 2} \mathrm{ft}$ above ground.

### 29.4 Exercise 29

1. Give an example of a sixth degree trinomial with leading coefficient 7. (The answer is not unique.)
2. Evaluate the sixth degree polynomial $f(x)=-2 x^{6}+5 x^{3}-5$ if $x=-1$.
3. A ball is thrown upward with velocity $\mathbf{1 0} \frac{\mathrm{ft}}{\mathrm{sec}}$ from a
platform 200 ft above ground. The equation of its path is $\boldsymbol{H}(\boldsymbol{t})=\mathbf{- 1 6} \boldsymbol{t}^{\mathbf{2}}+\mathbf{1 0 t}+\mathbf{2 0 0} \mathrm{ft}$.
How high above ground will the projectile be 4 seconds after launch?

## $S T \cap P!$

1. Give an example of a sixth degree trinomial with leading coefficient 7. (The answer is not unique.)

## Solution:

$\boldsymbol{7} \boldsymbol{x}^{\boldsymbol{6}}+\boldsymbol{a} \boldsymbol{x}^{\boldsymbol{m}}+\boldsymbol{b} \boldsymbol{x}^{\boldsymbol{n}}$ is such a polynomial. $\boldsymbol{a}$ can be any real number and $\boldsymbol{m}, \boldsymbol{n}$ are a whole number $\neq 6$ and $\boldsymbol{m} \neq \boldsymbol{n}$.
2. Evaluate the sixth degree polynomial $f(x)=-2 x^{6}+5 x^{3}-5$ if $x=-1$.

## Solution:

$$
\begin{aligned}
f(x) & =-2 x^{6}+5 x^{3}-5 \\
f(-1) & =-2(-1)^{6}+5(-1)^{3}-5 \\
& =-2(1)+5(-1)-5 \\
& =-17
\end{aligned}
$$

3. A ball is thrown upward with velocity $\mathbf{1 0} \frac{\mathrm{ft}}{\mathrm{sec}}$ from a platform $\mathbf{2 0 0} \mathrm{ft}$ above ground. The equation of its path is $H(t)=-\mathbf{1 6 t} t^{2}+\mathbf{1 0 t}+\mathbf{2 0 0} \mathrm{ft}$.
How high above ground will the projectile be 4 seconds after launch?

## Solution:

$$
\begin{aligned}
H(t) & =-16 t^{2}+10 t+200 \\
H(4) & =-16(4)^{2}+10(4)+200 \\
& =-16(16)+40+200 \\
& =-256+40+200 \\
& =-256+240 \\
& =-16
\end{aligned}
$$

The projectile will be $\mathbf{1 6} \mathrm{ft}$ below ground (if that is possible).

## Chapter 30

## Addition and Subtraction of Polynomials

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### 30.1 YouTube

https://www.youtube.com/playlist?list=PLD4235C06698DE3BD\&feature=view_all

### 30.2 Basics

A polynomial (in one variable) is a sum and/or difference of terms consisting exclusively of a special form. Each term is the product of a coefficient (constant number) and one (or more) variable(s) raised to non-negative integer.

Polynomials are written in order of decreasing (or increasing) exponents (called degrees). The degree of the polynomial is the highest exponent. (In case a term contains more than one variable factor, the degree of that term is the sum of the exponents of all the variables in that term).
$6 x^{7}+3 x^{3}-5$ is a polynomial of degree 7 .
$6 \boldsymbol{x}^{\mathbf{7}} \boldsymbol{y}^{\mathbf{3}}+\mathbf{3} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{\mathbf{2}}-\mathbf{5} \boldsymbol{y}$ is a polynomial of degree $\mathbf{1 0}$.
An equation of the height $s$ of a projectile being fired from a height $\mathbf{9} \mathrm{ft}$ above ground, with velocity 20
$\frac{\mathrm{ft}}{\mathrm{sec}}$ and acceleration $32 \frac{\mathrm{ft}}{\mathrm{sec}^{2}}$ is

$$
s=-16 t^{2}+20 t+9
$$

where $\boldsymbol{t}$ is the time (in seconds) after launching. The variable is $\boldsymbol{t}$. The polynomial is of degree $\mathbf{2}$.

Polynomials can be simplified if they contain like-terms. Two terms are like-terms if the variables are identical.
$\mathbf{6} \boldsymbol{x}^{\mathbf{3}}$ and $-\mathbf{4} \boldsymbol{x}^{\mathbf{3}}$ are like-terms. The sum of these like terms is $\mathbf{6} \boldsymbol{x}^{3}+(-4) x^{\mathbf{3}}=(6-4) \boldsymbol{x}^{\mathbf{3}}=\mathbf{2} \boldsymbol{x}^{\mathbf{3}}$.
Careful, not $2 x^{6}$.
Polynomials can be added and subtracted.

### 30.3 Examples

Example 1:
Add polynomials $P_{1}(x)=2 x^{5}+6 x^{3}-4 x+9$ to
$P_{2}(x)=8 x^{5}+7 x^{4}-6 x^{3}-4 x+12$

## Solution:

$$
\begin{aligned}
& P_{1}(x)+P_{2}(x) \\
= & \left(2 x^{5}+6 x^{3}-4 x+9\right)+\left(8 x^{5}+7 x^{4}-6 x^{3}-4 x+12\right) \\
= & \underline{2 x^{5}}+\underline{\underline{\underline{x^{3}}}}-\underbrace{4 x}+\underbrace{9}+\underline{x^{5}}+\underline{\underline{7 x^{4}}}-\underline{\underline{6 x}}^{9 x^{3}}-\underbrace{4 x}+\underbrace{12} \\
= & 2 x^{5}+8 x^{5}+7 x^{4}+6 x^{3}-6 x^{3}-4 x-4 x+9+12 \\
= & (2+8) x^{5}+7 x^{4}+(6-6) x^{3}+(-4-4) x+(9+12) \\
= & 10 x^{5}+7 x^{4}+(0) x^{3}+(-8) x+21 \quad \quad \begin{array}{l}
\text { underline like-terms } \\
\text { with the same symbols. }
\end{array} \\
= & 10 x^{5}+7 x^{4}-8 x+21 \quad \quad \begin{array}{l}
\text { wit }
\end{array}
\end{aligned}
$$

Example 2:
Subtract polynomials $P_{1}(x)=2 x^{5}+6 x^{3}-4 x+9$ from
$P_{2}(x)=8 x^{5}+7 x^{4}-6 x^{3}-4 x+12$

## Solution:

$$
\begin{aligned}
& P_{2}(x)-P_{1}(x) \\
= & \left(8 x^{5}+7 x^{4}-6 x^{3}-4 x+12\right)-\left(2 x^{5}+6 x^{3}-4 x+9\right) \\
& \text { "subtract from", } P_{2} \text { comes first } \\
= & 8 x^{5}+7 x^{4}-6 x^{3}-4 x+12-2 x^{5}-6 x^{3}+4 x-9
\end{aligned}
$$

Subtraction turns into addition of opposites.

$$
\begin{aligned}
& =\underline{8 x^{5}}+\underline{\underline{7 x^{4}}}-\underline{\underline{\underline{6 x^{3}}}}-\underbrace{4 x}+\underbrace{12}-\underline{2 x^{5}}-\underline{\underline{\underline{6 x^{3}}}}+\underbrace{4 x}-\underbrace{9} \\
& =8 x^{5}-2 x^{5}+7 x^{4}-6 x^{3}-6 x^{3}-4 x+4 x+12-9 \\
& =(8-2) x^{5}+7 x^{4}+(-6-6) x^{3}+(-4+4) x+3 \\
& =6 x^{5}+7 x^{4}-12 x^{3}+3
\end{aligned} \quad \begin{aligned}
& \text { underline like-terms } \\
& \text { with the same symbols. }
\end{aligned}
$$

Example 3:
Find $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ if

$$
a x^{4}+3 x^{2}+b x+9-\left(7 x^{4}+c x^{3}+d x^{2}+10 x+e\right)=-2 x^{4}+8 x^{3}+3 x^{2}-16 x
$$

## Solution:

$$
\begin{aligned}
& a x^{4}+3 x^{2}+b x+9-\left(7 x^{4}+c x^{3}+d x^{2}+10 x+e\right)=-2 x^{4}+8 x^{3}+3 x^{2}-16 x \\
& a x^{4}+3 x^{2}+b x+9-7 x^{4}-c x^{3}-d x^{2}-10 x-e=-2 x^{4}+8 x^{3}+3 x^{2}-16 x \\
& a x^{4}+\underline{\underline{\underline{x^{2}}}+\underbrace{b x}+\underbrace{9}-\underline{\underline{9} x^{4}}-\underline{\underline{c x^{3}}}-\underline{\underline{d x^{2}}}-\underbrace{10 x}-\underbrace{e}}=-\underline{\underline{2}}+\underline{\underline{4}}+\underline{\underline{\underline{s}}}-\underbrace{16 x} \\
& a x^{4}-7 x^{4}-c x^{3}+3 x^{2}-d x^{2}+b x-10 x+9-e=-2 x^{4}+8 x^{3}+3 x^{2}-16 x \\
&(a-7) x^{4}-c x^{3}+(3-d) x^{2}+(b-10) x+(9-e)=-2 x^{4}+8 x^{3}+3 x^{2}-16 x
\end{aligned}
$$

We deal with an equation. We need as many like-terms right as left of the equal sign.
For $x^{4}: a-7=-2$ or $a=7-2=5$.
For $x^{3}:-c=8$ or $c=-8$.
For $\boldsymbol{x}^{\mathbf{2}}: \mathbf{3}-\boldsymbol{d}=\mathbf{3}$ or $\boldsymbol{d}=\mathbf{0}$
For $x: b-10=-16$ or $b=10-16=-6$.
For the constants: $\mathbf{9}-\boldsymbol{e}=\mathbf{0}$ or $\boldsymbol{e}=\mathbf{9}$.
Are polynomials useful? You don't appreciate the importance yet. Each monomial $\boldsymbol{y}=\boldsymbol{x}^{\boldsymbol{m}}$ has a certain characteristic which is exploited graphically. A combination of polynomials is used to tweak a nice curve as an approximation to collected data approximation.

Go to the mall. Survey customers to give you their height and weight. Plot this data on a Cartesian coordinate system. Try to find a mathematical model (using monomials and graphing) that fits your data as closely as possible. You will handle elementary (linear) curve fitting in statistics (linear regression).

### 30.4 Exercise 30

1. Add polynomials $P_{1}(x)=3 x^{6}+7 x^{4}-5 x+10$ to

$$
P_{2}(x)=9 x^{6}+8 x^{5}-7 x^{4}-5 x+13
$$

2. Subtract polynomials $P_{1}(x)=3 x^{6}+7 x^{4}-5 x+10$ from

$$
P_{2}(x)=9 x^{6}+8 x^{5}-7 x^{4}-5 x+13
$$

3. Find $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ if

$$
a x^{5}+4 x^{3}+b x^{2}+10 x-\left(8 x^{5}+c x^{4}+d x^{3}+11 x^{2}+e x\right)
$$

$$
=-3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2}
$$

## STOP!

1. Add $P_{1}(x)=3 x^{6}+7 x^{4}-5 x+10$ to $P_{2}(x)=9 x^{6}+8 x^{5}-7 x^{4}-5 x+13$ Solution:

$$
P_{1}(x)+P_{2}(x)
$$

$=\left(3 x^{6}+7 x^{4}-5 x+10\right)+\left(9 x^{6}+8 x^{5}-7 x^{4}-5 x+13\right)$
$=\underline{3 x^{6}}+\underline{\underline{\underline{7 x^{4}}}}-\underbrace{5 x}+\underbrace{10}+\underline{9 x^{6}}+\underline{\underline{8 x^{5}}}-\underline{\underline{\underline{7 x^{4}}}}-\underbrace{5 x}+\underbrace{13}$
$=3 x^{6}+9 x^{6}+8 x^{5}+7 x^{4}-7 x^{4}-5 x-5 x+10+13$
$=(3+9) x^{6}+8 x^{5}+(7-7) x^{4}+(-5-5) x+(10+13)$
$=12 x^{6}+8 x^{5}+(0) x^{4}+(-10) x+23$
$=12 x^{6}+8 x^{5}-10 x+23$
2. Subtract polynomials $P_{1}(x)=3 x^{6}+7 x^{4}-5 x+10$ from $P_{2}(x)=9 x^{6}+8 x^{5}-7 x^{4}-5 x+13$

## Solution:

$$
\begin{aligned}
& P_{2}(x)-P_{1}(x) \\
= & \left(9 x^{6}+8 x^{5}-7 x^{4}-5 x+13\right)-\left(3 x^{6}+7 x^{4}-5 x+10\right) \\
= & 9 x^{6}+8 x^{5}-7 x^{4}-5 x+13-3 x^{6}-7 x^{4}+5 x-10 \\
= & \underline{9 x^{6}}+\underline{\underline{8 x^{5}}}-\underline{\underline{7 x^{4}}}-\underbrace{5 x}+\underbrace{13}-\underline{x x}^{3 x^{6}}-\underline{\underline{\underline{7 x^{4}}}}+\underbrace{5 x}-\underbrace{10} \\
= & \underline{9 x^{6}}-\underline{3 x^{6}}+\underline{\underline{8 x^{5}}}-\underline{\underline{7 x^{4}}}-\underline{\underline{7 x^{4}}}-\underbrace{5 x}+\underbrace{5 x}+\underbrace{13}-\underbrace{10} \\
= & 9 x^{6}-3 x^{6}+8 x^{5}-7 x^{4}-7 x^{4}-5 x+5 x+13-10 \\
= & (9-3) x^{6}+8 x^{5}+(-7-7) x^{4}+(-5+5) x+3 \\
= & 6 x^{3}+8 x^{5}-14 x^{4}+3
\end{aligned}
$$

3. Find $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ if
$a x^{5}+4 x^{3}+b x^{2}+10 x-\left(8 x^{5}+c x^{4}+d x^{3}+11 x^{2}+e x\right)=-3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2}$
Solution:

$$
\begin{aligned}
& a x^{5}+4 x^{3}+b x^{2}+10 x-\left(8 x^{5}+c x^{4}+d x^{3}+11 x^{2}+e x\right)=3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2} \\
& a x^{5}+4 x^{3}+b x^{2}+10 x-8 x^{5}-c x^{4}-d x^{3}-11 x^{2}-e x=-3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2} \\
& \underline{a x^{5}}+\underline{\underline{\underline{4 x^{3}}}}+\underbrace{b x^{2}}+\underbrace{10 x}-\underline{8 x^{5}}-\underline{\underline{c x^{4}}}-\underline{\underline{\underline{d x^{3}}}}-\underbrace{11 x^{2}}-\underbrace{\underbrace{e x}}=-\underline{3 x^{5}}+\underline{\underline{9 x^{4}}}+\underline{\underline{\underline{4 x^{3}}}}-\underbrace{14 x^{2}} \\
& \underline{a x^{5}}-\underline{8 x^{5}}-\underline{\underline{c x^{4}}}+\underline{\underline{\underline{4 x^{3}}}}-\underline{\underline{\underline{d x^{3}}}}+\underbrace{b x^{2}}-\underbrace{11 x^{2}}+\underbrace{10 x}-\underbrace{e x}=-\underline{\underline{x^{5}}}+\underline{\underline{9 x^{4}}}+\underline{\underline{\underline{x^{3}}}}-\underbrace{14 x^{2}} \\
& a x^{5}-8 x^{5}-c x^{4}+4 x^{3}-d x^{3}+b x^{2}-11 x^{2}+10 x-e x=-3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2} \\
& (a-8) x^{5}-c x^{4}+(4-d) x^{3}+(b-11) x^{2}+(10-e) x=-3 x^{5}+9 x^{4}+4 x^{3}-14 x^{2}
\end{aligned}
$$

Since we deal with an equation, we need as many like-terms on the right as on the left of the equal sign.
For $x^{5}: a-8=-3$ or $a=8-3=5$.
For $x^{4}:-c=9$ or $c=-9$.
For $x^{3}: 4-d=4$ or $d=0$
For $x^{2}: b-11=-14$ or $b=11-14=-3$.
For $\boldsymbol{x}: \mathbf{1 0}-\boldsymbol{e}=\mathbf{0}$ or $\boldsymbol{e}=\mathbf{1 0}$.

## Chapter 31

## Multiplication of Polynomials

(c) H. Feiner 2011

### 31.1 YouTube

https://www.youtube.com/playlist?list=PL379A738B390FB543\&feature=view_all

### 31.2 Basics

Remember that

$$
2^{3} \cdot 2^{4}=(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)=2^{3+4}=2^{7}
$$

Thus

$$
a^{m} a^{n}=a^{m+n}
$$

$\left(2^{3}\right)^{4}=(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)=2^{3 \cdot 4}=2^{12}$
Thus

$$
\left(a^{m}\right)^{n}=a^{m n}
$$

### 31.3 Examples

## Example 1:

Simplify $\left(2 x^{3} \boldsymbol{y}^{4}\right)\left(-6 x^{5} y\right)^{2}$

## Solution:

$$
\begin{aligned}
\left(2 x^{3} y^{4}\right)\left(-6 x^{5} y\right)^{2} & =\left(2 x^{3} y^{4}\right)(-6)^{2}\left(x^{5}\right)^{2}(y)^{2} \\
& =\left(2 x^{3} y^{4}\right)(36) x^{10} y^{2} \\
& =72 x^{3+10} y^{4+2} \\
& =72 x^{13} y^{6}
\end{aligned}
$$

Note: Even if you do not remember whether to add or multiply exponents, the math is talking to you:

$$
\begin{aligned}
& \left(2 x^{3} y^{4}\right)\left(-6 x^{5} y\right)^{2} \\
& =\left(2 x^{3} y^{4}\right)\left(-6 x^{5} y\right)\left(-6 x^{5} y\right) \\
& \left.=(2)(-6)(-6)(x x x y y y y)\left(\begin{array}{c}
x
\end{array}\right) x x x y\right)(x x x x x y) \\
& =72 x^{13} y^{6}
\end{aligned}
$$

Example 2:
Simplify and leave no negative exponents:
$\left(\frac{-2^{4} x^{-3} y^{5} z}{4^{2} x^{6} y^{-5} z^{4}}\right)^{-3}$

## Solution:

$$
\begin{aligned}
\left(\frac{-2^{4} x^{-3} y^{5} z}{4^{2} x^{6} y^{-5} z^{4}}\right)^{-3} & =\left(\frac{-16 x^{-3-6} y^{5+5} z^{1-4}}{16}\right)^{-3} \\
& =\left(\frac{-x^{-9} y^{10} z^{-3}}{1}\right)^{-3} \\
& =\frac{(-1)^{-3} x^{(-9)(-3)} y^{(10)(-3)} z^{(-3)(-3)}}{1} \\
& =\frac{-x^{27} y^{-30} z^{9}}{1} \\
& =-\frac{x^{27} z^{9}}{y^{30}}
\end{aligned}
$$

Example 3:
Simplify and leave no negative exponents:
$\frac{x^{-2} y^{-3}}{z^{-4}}$

## Solution:

$$
\frac{x^{-2} y^{-3}}{z^{-4}}=\frac{z^{4}}{x^{2} y^{3}}
$$

Example 4:
Simplify and leave no negative exponents:

$$
\frac{x^{-2}+y^{-3}}{z^{-4}}
$$

## Solution:

$$
\begin{aligned}
\frac{x^{-2}+y^{-3}}{z^{-4}} & =\frac{\frac{1}{x^{2}}+\frac{1}{y^{3}}}{\frac{1}{z^{4}}} \\
& =\frac{\frac{y^{3}}{x^{2} y^{3}}+\frac{x^{2}}{x^{2} y^{3}}}{\frac{1}{z^{4}}} \\
& =\frac{\frac{y^{3}+x^{2}}{x^{2} y^{3}}}{\frac{1}{z^{4}}} \\
& =\frac{\left(y^{3}+x^{2}\right) z^{4}}{x^{2} y^{3}}
\end{aligned}
$$

## Example 5:

Find the product $(2 x-5)\left(x^{3}-x^{2}+3 x-4\right)$

## Solution:

$$
\begin{aligned}
& (2 x-5)\left(x^{3}-x^{2}+3 x-4\right) \\
= & 2 x\left(x^{3}-x^{2}+3 x-4\right)-5\left(x^{3}-x^{2}+3 x-4\right) \\
= & 2 x\left(x^{3}\right)-2 x\left(x^{2}\right)+2 x(3 x)-2 x(4)-5\left(x^{3}\right)-(-5) x^{2}+(-5) 3 x-5(-4) \\
= & 2 x^{4}-2 x^{3}+6 x^{2}-8 x-5 x^{3}+5 x^{2}-15 x+20
\end{aligned}
$$

Example 6:
Multiply $\left(3 x^{2}+4\right)\left(3 x^{2}-4\right)$

## Solution:

$$
\begin{aligned}
\left(3 x^{2}+4\right)\left(3 x^{2}-4\right) & =3 x^{2}\left(3 x^{2}-4\right)+4\left(3 x^{2}-4\right) \\
& =3 x^{2}\left(3 x^{2}\right)-4\left(3 x^{2}\right)+4\left(3 x^{2}\right)-(4) 4 \\
& =\left(3 x^{2}\right)^{2}-12 x^{2}+12 x^{2}-4^{2} \\
& =9 x^{4}-16
\end{aligned}
$$

We shall develop and remember the identity

$$
(a+b)(a-b)=a^{2}-b^{2}
$$

and memorize that

## Example 7:

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

Multiply (expand) $\left(3 x^{2}+4\right)^{2}$

## Solution:

$$
\begin{aligned}
\left(3 x^{2}+4\right)^{2} & =\left(3 x^{2}+4\right)\left(3 x^{2}+4\right) \\
& =3 x^{2}\left(3 x^{2}+4\right)+4\left(3 x^{2}+4\right) \\
& =3 x^{2}\left(3 x^{2}\right)+4\left(3 x^{2}\right)+4\left(3 x^{2}\right)-(4) 4 \\
& =\left(3 x^{2}\right)^{2}+2(12) x^{2}+4^{2} \text { Do not forget the } \\
& =\left(3 x^{2}\right)^{2}+24 x^{2}+4^{2} \quad \text { double product term } \\
& =9 x^{4}+24 x^{2}+16
\end{aligned}
$$

Don't confuse squaring a binomial with squaring a product $\left[\left(3 x^{2}\right)(4)\right]^{2}=\left(9 x^{4}\right)(16)$

### 31.4 Exercise 31

1. Simplify

$$
\left(4 x^{6} y^{3}\right)^{2}\left(-2^{2} x^{4} y^{2}\right)
$$

2. Simplify and leave no negative exponents:

$$
\left(\frac{-3^{4} x^{-4} y^{6} z}{9^{2} x^{2} y^{-3} z^{5}}\right)^{-2}
$$

3. Simplify and leave no negative exponents:

$$
\frac{x^{-5} y^{-1}}{z^{-7}}
$$

4. Simplify and leave no negative exponents:

$$
\frac{x^{-5}+y^{-1}}{z^{-7}}
$$

5. Find the product

$$
\left(3 x^{2}-4 x+1\right)\left(x^{2}-4 x+5\right)
$$

6. Multiply

$$
\left(5 x^{3}+6\right)\left(5 x^{3}-6\right)
$$

7. Multiply (expand)

$$
\left(5 x^{3}+6\right)^{2}
$$

## STOP!

1. Simplify $\left(4 x^{6} y^{3}\right)^{2}\left(-2^{2} x^{4} y^{2}\right)$

## Solution:

$$
\begin{aligned}
\left(4 x^{6} y^{3}\right)^{2}\left(-2^{2} x^{4} y^{2}\right) & =4^{2}\left(x^{6}\right)^{2}\left(y^{3}\right)^{2}\left(-2^{2} x^{4} y^{2}\right) \\
& =-16\left(x^{12}\right)\left(y^{6}\right)\left(4 x^{4} y^{2}\right) \\
& =-64 x^{12+4} y^{6+2} \\
& =-64 x^{16} y^{8}
\end{aligned}
$$

Note: Even if you do not remember whether to add or multiply exponents, the math is talking to you:

$$
\begin{aligned}
& \left(4 x^{6} y^{3}\right)^{2}\left(-2^{2} x^{4} y^{2}\right) \\
= & \left(4 x^{6} y^{3}\right)\left(4 x^{6} y^{3}\right)\left(-2^{2} x^{4} y^{2}\right) \\
= & (4 x x x x x x x y y y y)(4 x x x x x x y y y)(-2 \cdot 2 x x x x y y) \\
= & -64 x^{16} y^{8}
\end{aligned}
$$

2. Simplify and leave no negative exponents:
$\left(\frac{-3^{4} x^{-4} y^{6} z}{9^{2} x^{2} y^{-3} z^{5}}\right)^{-2}$
Solution:

$$
\begin{aligned}
\left(\frac{-3^{4} x^{-4} y^{6} z}{9^{2} x^{2} y^{-3} z^{5}}\right)^{-2} & =\frac{\left(-3^{4}\right)^{-2}\left(x^{-4}\right)^{-2} y^{6(-2)} z^{-2}}{\left(9^{2}\right)^{-2}\left(x^{2}\right)^{-2}\left(y^{-3}\right)^{-2}\left(z^{5}\right)^{-2}} \\
& =\frac{3^{4(-2)} x^{(-4)(-2)} y^{6(-2)} z^{-2}}{9^{2(-2)} x^{2(-2)} y^{(-3)(-2)} z^{5(-2)}} \\
& =\frac{3^{-8} x^{8} y^{-12} z^{-2}}{9^{-4} x^{-4} y^{6} z^{-10}} \\
& =\frac{3^{-8} x^{8+4} z^{-2+10}}{\left(3^{2}\right)^{-4} y^{6+12}} \\
& =\frac{3^{-8} x^{12} z^{8}}{3^{-8} y^{18}} \\
& =\frac{x^{12} z^{8}}{y^{18}}
\end{aligned}
$$

3. Simplify and leave no negative exponents:
$\frac{x^{-5} y^{-1}}{z^{-7}}$
Solution:

$$
\frac{x^{-5} y^{-1}}{z^{-7}}=\frac{z^{7}}{x^{5} y}
$$

4. Simplify and leave no negative exponents:
$\frac{x^{-5}+y^{-1}}{z^{-7}}$
Solution:

$$
\begin{aligned}
\frac{x^{-5}+y^{-1}}{z^{-7}} & =\frac{\frac{1}{x^{5}}+\frac{1}{y}}{\frac{1}{z^{7}}} \\
& =\frac{\frac{y}{x^{5} y}+\frac{x^{5}}{x^{5} y}}{\frac{1}{z^{7}}} \\
& =\frac{\left(x^{5}+y\right) z^{7}}{x^{5} y}
\end{aligned}
$$

5. Find the product $\left(3 x^{2}-4 x+1\right)\left(x^{2}-4 x+5\right)$

Solution:

$$
\begin{aligned}
& \left(3 x^{2}-4 x+1\right)\left(x^{2}-4 x+5\right) \\
= & 3 x^{2}\left(x^{2}-4 x+5\right)-4 x\left(x^{2}-4 x+5\right)+\left(x^{2}-4 x+5\right) \\
= & 3 x^{2}\left(x^{2}\right)-3 x^{2}(4 x)+\left(3 x^{2}\right) 5-4 x\left(x^{2}\right)-(-4 x) 4 x+(-4 x) 5+x^{2}-4 x+5 \\
= & 3 x^{4}-12 x^{3}+15 x^{2}-4 x^{3}+16 x^{2}-20 x+x^{2}-4 x+5 \\
= & 3 x^{4}-12 x^{3}-4 x^{3}+15 x^{2}+16 x^{2}+x^{2}-20 x-4 x+5 \\
= & 3 x^{4}-16 x^{3}+32 x^{2}-24 x+5
\end{aligned}
$$

6. Multiply $\left(5 x^{3}+6\right)\left(5 x^{3}-6\right)$

## Solution:

$$
\begin{aligned}
\left(5 x^{3}+6\right)\left(5 x^{3}-6\right) & =5 x^{3}\left(5 x^{3}-6\right)+6\left(5 x^{3}-6\right) \\
& =5 x^{3}\left(5 x^{3}\right)-6\left(5 x^{3}\right)+6\left(5 x^{3}\right)-(6) 6 \\
& =\left(5 x^{3}\right)^{2}-30 x^{3}+30 x^{3}-6^{2} \\
& =25 x^{6}-36
\end{aligned}
$$

7. Multiply (expand) $\left(5 x^{3}+6\right)^{2}$

## Solution:

$$
\begin{aligned}
\left(5 x^{3}+6\right)^{2} & =\left(5 x^{3}+6\right)\left(5 x^{3}+6\right) \\
& =5 x^{3}\left(5 x^{3}+6\right)+6\left(5 x^{3}+6\right) \\
& =5 x^{3}\left(5 x^{3}\right)+6\left(5 x^{3}\right)+6\left(5 x^{3}\right)+(6) 6 \\
& =5 x^{3}\left(5 x^{3}\right)+30 x^{3}+30 x^{3}+(6) 6 \\
& =\left(5 x^{3}\right)^{2}+2(30) x^{3}+6^{2} \\
& =\left(5 x^{3}\right)^{2}+60 x^{3}+6^{2} \\
& =25 x^{6}+60 x^{3}+36
\end{aligned}
$$

Do not forget the double product term $2\left(5 \boldsymbol{x}^{2}\right)(6)$.

## Chapter 32

## Special Products

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### 32.1 YouTube

https://www.youtube.com/playlist?list=PLC6CDB9A1EE6FD9B6\&feature=view_all

### 32.2 Basics

1. $(a+b)(a-b)=a^{2}-b^{2}$
2. $(a+b)^{2}=a^{2}+2 a b+b^{2}$
3. $(a-b)^{2}=a^{2}-2 a b+b^{2}$
4. $(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}$ (optional)
5. $(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3}$ (optional)
6. $a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)$ (optional)
7. $a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right)$ (optional)

Development of the formulas above:

1. $(a+b)(a-b)=a(a-b)+b(a-b)$

$$
\begin{aligned}
& =a^{2}-a b+a b-b^{2} \\
& =a^{2}-b^{2}
\end{aligned}
$$

2. $(a+b)^{2}=(a+b)(a+b)$

$$
\begin{aligned}
& =a(a+b)+b(a+b) \\
& =a^{2}+a b+a b+b^{2} \\
& =a^{2}+2 a b+b^{2}
\end{aligned}
$$

3. $(a-b)^{2}=(a-b)(a-b)$

$$
\begin{aligned}
& =a(a-b)-b(a-b) \\
& =a^{2}-a b-a b+b^{2} \\
& =a^{2}-2 a b+b^{2}
\end{aligned}
$$

4. $(a+b)^{3}=(a+b)(a+b)^{2}$

$$
\begin{aligned}
& =a(a+b)^{2}+b(a+b)^{2} \\
& =a\left(a^{2}+2 a b+b^{2}\right)+b\left(a^{2}+2 a b+b^{2}\right) \\
& =a^{3}+2 a^{2} b+a b^{2}+a^{2} b+2 a b^{2}+b^{3} \\
& =a^{3}+2 a^{2} b+a^{2} b+a b^{2}+2 a b^{2}+b^{3} \\
& =a^{3}+3 a^{2} b+3 a b^{2}+b^{3}
\end{aligned}
$$

5. $(a-b)^{3}=(a-b)(a-b)^{2}$

$$
\begin{aligned}
& =a(a-b)^{2}-b(a-b)^{2} \\
& =a\left(a^{2}-2 a b+b^{2}\right)-b\left(a^{2}-2 a b+b^{2}\right) \\
& =a^{3}-2 a^{2} b+a b^{2}-a^{2} b+2 a b^{2}-b^{3} \\
& =a^{3}-2 a^{2} b-a^{2} b+a b^{2}+2 a b^{2}-b^{3} \\
& =a^{3}-3 a^{2} b+3 a b^{2}-b^{3}
\end{aligned}
$$

6. $(a-b)\left(a^{2}+a b+b^{2}\right)=a\left(a^{2}+a b+b^{2}\right)-b\left(a^{2}+a b+b^{2}\right)$

$$
\begin{aligned}
& =a^{3}+a^{2} b+a b^{2}-a^{2} b-a b^{2}-b^{3} \\
& =a^{3}+a^{2} b-a^{2} b+a b^{2}-a b^{2}-b^{3} \\
& =a^{3}-b^{3}
\end{aligned}
$$

7. $(a+b)\left(a^{2}-a b+b^{2}\right)=a\left(a^{2}-a b+b^{2}\right)+b\left(a^{2}-a b+b^{2}\right)$

$$
\begin{aligned}
& =a^{3}-a^{2} b+a b^{2}+a^{2} b-a b^{2}+b^{3} \\
& =a^{3}-a^{2} b+a^{2} b+a b^{2}-a b^{2}+b^{3} \\
& =a^{3}+b^{3}
\end{aligned}
$$

In identities $\mathbf{4}$ and $\mathbf{5}$ note the decrease in exponents $\boldsymbol{a}^{\mathbf{3}}, \boldsymbol{a}^{\mathbf{2}}, \boldsymbol{a}, \boldsymbol{a}^{\mathbf{0}}$ and the increase in the exponents $\boldsymbol{b}^{\mathbf{0}}, \boldsymbol{b}, \boldsymbol{b}^{\mathbf{2}}, \boldsymbol{b}^{\mathbf{3}}$ from term to term.

### 32.3 Examples

## Example 1:

Multiply

$$
\left(7 x+y^{2}\right)\left(7 x-y^{2}\right)
$$

Solution:

$$
\begin{aligned}
(a+b)(a-b) & =a^{2}-b^{2} \\
\left(7 x+y^{2}\right)\left(7 x-y^{2}\right) & =(7 x)^{2}-\left(y^{2}\right)^{2} \quad a=7 x, \quad b=y^{2} \\
& =49 x^{2}-y^{4}
\end{aligned}
$$

Example 2:
Multiply (expand)

$$
\left[\left(7 x+y^{2}\right)-5\right]\left[\left(7 x+y^{2}\right)+5\right]
$$

Solution:

$$
\begin{aligned}
& (a+b)(a-b)=a^{2}-b^{2} \text { identity } 1 \\
& (a+b)^{2}=a^{2}+2 a b+b^{2} \text { identity } 2 \\
& {\left[\left(7 x+y^{2}\right)-5\right]\left[\left(7 x+y^{2}\right)+5\right]} \\
& =\left(7 x+y^{2}\right)^{2}-5^{2} \quad a=\left(7 x+y^{2}\right), \quad b=5 \quad \text { identity } 1 \\
& =(7 x)^{2}+2 \cdot 7 x y^{2}+\left(y^{2}\right)^{2}-25 \quad a=7 x, \quad b=y^{2} \quad \text { identity } 2 \\
& =49 x^{2}+14 x y^{2}+y^{4}-25
\end{aligned}
$$

## Example 3:

Find $\boldsymbol{k}$ such that $\boldsymbol{k}^{2} \boldsymbol{x}^{2}-\mathbf{2 4 x}+\mathbf{3 6}$ is a perfect square.

## Solution:

$$
\begin{aligned}
(a-b)^{2}= & a^{2}-2 a b+b^{2} \\
& a^{2}-2 a b+b^{2}
\end{aligned} \| \begin{aligned}
& k^{2} x^{2}-24 x+36 \\
& (k x)^{2}-2(k x)(6)+6^{2}
\end{aligned}
$$

$2(k x)(6)=24 x$ so $12 k=24$ or $k=2$.
The perfect square is $(2 x-6)^{2}$.
Check:

$$
\begin{aligned}
(2 x-6)^{2} & =(2 x)^{2}-2(2 x)(6)+6^{2} \\
& =4 x^{2}-24 x+36
\end{aligned}
$$

Example 4:
Expand $(2 x+5)^{3}$.
Solution:

$$
\begin{aligned}
& (a+b)^{3}=a^{3}+3 a^{2} b+a b^{2}+b^{3} \\
& \begin{aligned}
(2 x+5)^{3} & =(2 x)^{3}+3(2 x)^{2}(5)+3(2 x)(5)^{2}+(5)^{3} \\
& =8 x^{3}+60 x^{2}+150 x+125
\end{aligned}
\end{aligned}
$$

## Example 5:

Find $c$ such that $(4 x-5)^{3}=64 x^{3}-48 c x^{2}+12 c^{2} x-c^{3}$.
Solution:

Thus $\boldsymbol{c}=5$.
Example 6:
Simplify without multiplying $(9 x \pm 4 y)\left(81 x^{2} \mp 36 x y+16 y^{2}\right)$

## Solution:

$$
\begin{aligned}
& (9 x \pm 4 y)\left(81 x^{2} \mp 36 x y+16 y^{2}\right) \\
= & (9 x \pm 4 y)\left[(9 x)^{2} \mp(9 x)(4 y)+(4 y)^{2}\right] \\
= & (9 x)^{3} \pm(4 y)^{3} \\
= & 729 x^{3} \pm 64 y^{3}
\end{aligned}
$$

### 32.4 Exercise 32

1. Multiply

$$
\left(5 y^{2}+6 z\right)\left(5 y^{2}-6 z\right)
$$

2. Multiply (expand)

$$
\left[\left(5 y^{2}+6 z\right)-7\right]\left[\left(5 y^{2}+6 z\right)+7\right]
$$

3. Find $\boldsymbol{k}$ such that $\mathbf{1 6} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{-} \boldsymbol{k} \boldsymbol{x}+\mathbf{8 1}$ is a perfect square.
4. Expand $\left(\mathbf{3} y+5 z^{2}\right)^{3}$.
5. Find $c$ such that $(c x-2)^{3}=64 x^{3}-96 x^{2}+48 x-8$.
6. Simplify without multiplying

$$
(5 x \pm 3 y)\left(25 x^{2} \mp 15 x y+9 y^{2}\right)
$$

## STOP!

1. Multiply
$\left(5 y^{2}+6 z\right)\left(5 y^{2}-6 z\right)$
Solution:

$$
\begin{aligned}
& (a+b)(a-b)=a^{2}-b^{2} \\
& \begin{aligned}
\left(5 y^{2}+6 z\right)\left(5 y^{2}-6 z\right) & =\left(5 y^{2}\right)^{2}-(6 z)^{2} \quad a=5 y^{2}, \quad b=6 z^{2} \\
& =25 y^{4}-36 z^{2}
\end{aligned}
\end{aligned}
$$

2. Multiply (expand)

$$
\left[\left(5 y^{2}+6 z\right)-7\right]\left[\left(5 y^{2}+6 z\right)+7\right]
$$

Solution:

$$
\begin{aligned}
& (a-b)(a+b)=a^{2}-b^{2} \\
& (a+b)^{2}=a^{2}+2 a b+b^{2} \\
& {\left[\left(5 y^{2}+6 z\right)-7\right]\left[\left(5 y^{2}+6 z\right)+7\right]} \\
& =\left(5 y^{2}+6 z\right)^{2}-7^{2} \quad a=5 y^{2}+6 z, b=7 \text { identity } 1 \\
& =\left(5 y^{2}\right)^{2}+60 y^{2} z+(6 z)^{2}-49 \quad a=5 y^{2}, \quad b=6 z \text { identity } 2 \\
& =25 y^{4}+60 y^{2} z+36 z^{2}-49
\end{aligned}
$$

3. Find $\boldsymbol{k}$ such that $\mathbf{1 6} \boldsymbol{x}^{\mathbf{2}}-\boldsymbol{k} \boldsymbol{x}+\mathbf{8 1}$ is a perfect square.

Solution:

$$
\begin{aligned}
(a-b)^{2}= & a^{2}-2 a b+b^{2} \| \\
& a^{2}-2 a b+b^{2}
\end{aligned} \| \begin{aligned}
& 16 x^{2}-k x+81 \\
& (4 x)^{2}-2(4 x)(9)+9^{2}
\end{aligned}
$$

$k x=(2)(4 x)(9)$ so $k=72$.
4. Expand $\left(\mathbf{3} y+5 z^{2}\right)^{3}$.

## Solution:

$$
\begin{aligned}
&(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3} \\
&\left(3 y+5 z^{2}\right)^{3} \\
&=(3 y)^{3}+3(3 y)^{2}\left(5 z^{2}\right)+3(3 y)\left(5 z^{2}\right)^{2}+\left(5 z^{2}\right)^{3} \\
&= 27 y^{3}+135 y^{2} z^{2}+225 y z^{4}+125 z^{6}
\end{aligned}
$$

5. Find $c$ such that $(c x-2)^{3}=64 x^{3}-96 x^{2}+48 x-8$.

## Solution:

$$
\begin{array}{l|l}
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
(c x-2)^{3}=(c x)^{3}-3(c x)^{2}(2)+3(c x)(2)^{2}+2^{3}
\end{array} \quad \begin{aligned}
& 64 x^{3}-96 x^{2}+48 x-8 \\
& (4 x-2)^{3}=64 x^{3}-3(4 x)^{2}(2)+3(4 x)(4)-8
\end{aligned}
$$

Thus $c=2$.
6. Simplify without multiplying

$$
(5 x \pm 3 y)\left(25 x^{2} \mp 15 x y+9 y^{2}\right)
$$

## Solution:

$$
\begin{aligned}
& (5 x \pm 3 y)\left(25 x^{2} \mp 15 x y+9 y^{2}\right) \\
= & (5 x \pm 3 y)\left[(5 x)^{2} \mp(5 x)(3 y)+(3 y)^{2}\right] \|(a \pm b)\left[a^{2} \mp a b+b^{2}\right]=a^{3} \pm b^{3} \\
= & (5 x)^{3} \pm(3 y)^{3}
\end{aligned}
$$

## Chapter 33

## Polynomials in several Variables

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### 33.1 YouTube

https://www.youtube.com/playlist?list=PLE8DBED66E6BEC03B\&feature=view_all

### 33.2 Basics

As the title of this chapter implies, the polynomials have more than one variable.
$6 x^{7} y^{3}-5 x^{4} y+x^{2}-y+9$
is such a polynomial. The degree of each term is the sum of the number of variables in that term. The first term in the polynomial above is of degree $\mathbf{7 + 3}=\mathbf{1 0}$. Like-terms are terms in which the variables are identical.
$5 \boldsymbol{x}^{4} \boldsymbol{y}^{\mathbf{3}}-\mathbf{7} \boldsymbol{x}^{4} \boldsymbol{y}^{\mathbf{3}}=-\mathbf{2} \boldsymbol{x}^{4} \boldsymbol{y}^{\mathbf{3}}$ because the terms are like-terms.

### 33.3 Examples

Example 1:
Simplify $2 x^{3} y+3 x^{2} y^{2}-4 x y^{4}+5 x^{3} y-6 x^{2} y^{2}+7 x y^{4}$

## Solution:

$$
\begin{aligned}
& 2 x^{3} y+3 x^{2} y^{2}-4 x y^{4}+5 x^{3} y-6 x^{2} y^{2}+7 x y^{4} \\
= & 2 x^{3} y+5 x^{3} y+3 x^{2} y^{2}-6 x^{2} y^{2}-4 x y^{4}+7 x y^{4} \\
= & 7 x^{3} y-3 x^{2} y^{2}+3 x y^{4}
\end{aligned}
$$

Example 2:
Evaluate $\frac{x^{3}-y^{3}}{x^{2}+x y+y^{2}}$ if $x=4$ and -3 .
Solution:

$$
\begin{aligned}
\frac{x^{3}-y^{3}}{x^{2}+x y+y^{2}} & =\frac{4^{3}-(-3)^{3}}{4^{2}+4(-3)+(-3)^{2}} \\
& =\frac{64-(-27)}{16-12+9} \\
& =\frac{64+27}{4+9} \\
& =\frac{91}{13} \\
& =7
\end{aligned}
$$

If you studied $\boldsymbol{a}^{\mathbf{3}} \boldsymbol{-} \boldsymbol{b}^{\mathbf{3}}$ under special products, then do you see why the answer is $4-(-3)=7$ ?

Example 3:
Find the sum of $P_{1}=6 r^{5} t^{3}+7 r^{3} t^{2}-8 r t^{6}$ and $P_{2}=9 r^{5} t^{3}+10 r^{2} t^{3}+11 r t^{6}$

## Solution:

$$
\begin{aligned}
P_{1}+P_{2} & =6 r^{5} t^{3}+7 r^{3} t^{2}-8 r t^{6}+9 r^{5} t^{3}+10 r^{2} t^{3}+11 r t^{6} \\
& =6 r^{5} t^{3}+9 r^{5} t^{3}+7 r^{3} t^{2}+10 r^{2} t^{3}-8 r t^{6}+11 r t^{6} \\
& =15 r^{5} t^{3}+7 r^{3} t^{2}+10 r^{2} t^{3}+3 r t^{6}
\end{aligned}
$$

Example 4:
Find the difference of $P_{1}=6 r^{5} t^{3}+7 r^{3} t^{2}-8 r t^{6}$ and $P_{2}=9 r^{5} t^{3}+10 r^{2} t^{3}+11 r t^{6}$
Solution:

$$
\begin{aligned}
P_{1}-P_{2} & =6 r^{5} t^{3}+7 r^{3} t^{2}-8 r t^{6}-\left(9 r^{5} t^{3}+10 r^{2} t^{3}+11 r t^{6}\right) \\
& =6 r^{5} t^{3}+7 r^{3} t^{2}-8 r t^{6}-9 r^{5} t^{3}-10 r^{2} t^{3}-11 r t^{6} \\
& =6 r^{5} t^{3}-9 r^{5} t^{3}+7 r^{3} t^{2}-10 r^{2} t^{3}-8 r t^{6}-11 r t^{6} \\
& =-3 r^{5} t^{3}+7 r^{3} t^{2}-10 r^{2} t^{3}-19 r t^{6}
\end{aligned}
$$

Note: If the problem had stated "subtract $\boldsymbol{P}_{\mathbf{1}}$ from $\boldsymbol{P}_{\mathbf{2}}$ " then the subtraction would have been

$$
P_{2}-P_{1}=3 r^{5} t^{3}-7 r^{3} t^{2}+10 r^{2} t^{3}+19 r t^{6}
$$

Example 5
Find the product of $\boldsymbol{P}_{\mathbf{3}}=\mathbf{5} \boldsymbol{x}^{2} \boldsymbol{y}^{\mathbf{3}}+\mathbf{4 x} \boldsymbol{y}^{6}$ and $\boldsymbol{P}_{\mathbf{4}}=5 \boldsymbol{x}^{2} \boldsymbol{y}^{\mathbf{3}}-\mathbf{4} \boldsymbol{x} \boldsymbol{y}^{\mathbf{6}}$
Solution:

$$
\begin{aligned}
P_{3} P_{4} & =\left(5 x^{2} y^{3}+4 x y^{6}\right)\left(5 x^{2} y^{3}-4 x y^{6}\right) \\
& =\left(5 x^{2} y^{3}\right)^{2}-\left(4 x y^{6}\right)^{2} \\
& =25 x^{4} y^{6}-16 x^{2} y^{12}
\end{aligned}
$$

Example 6:
Find the quotient of $\boldsymbol{P}_{\mathbf{5}}=\mathbf{1 5} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}^{\mathbf{3}}$ and $\boldsymbol{P}_{\mathbf{6}}=\mathbf{- 3 \boldsymbol { x }} \boldsymbol{y}^{\mathbf{6}}$. Leave no negative exponents.
Solution:

$$
\begin{aligned}
\frac{P_{5}}{P_{6}} & =\frac{15 x^{2} y^{3}}{-3 x y^{6}} \\
& =-5 x^{2-1} y^{3-6} \\
& =-\frac{5 x}{y^{3}}
\end{aligned}
$$

Example 7:
A can of soda is a right circular cylinder with top and bottom circles of radius $\boldsymbol{r}$, and cylinder height $\boldsymbol{h}$.
(a) Find the surface area, a polynomial in the two variables $\boldsymbol{r}$ and $\boldsymbol{h}$.
(b) Find the volume of the can.

## Solution:

The top and bottom have area $\left.\boldsymbol{A}_{\boldsymbol{t}}=\mathbf{2 (} \boldsymbol{\pi} \boldsymbol{r}^{2}\right)$.
(a) To find the lateral surface area slit the surface perpendicular to the base. Flatten the lateral surface to form a rectangle. The height of the rectangle is the height $\boldsymbol{h}$ of the cylinder. The base of the rectangle is the circumference of the top (or bottom) circle, $C=\mathbf{2 \pi r}$.
The total surface area
$=2 \pi r^{2}+2 \pi r h$.
(b) The volume is base times height $\boldsymbol{V}=\boldsymbol{\pi} \boldsymbol{r}^{2} \boldsymbol{h}$.


### 33.4 Exercise 33

1. Simplify $3 x^{4} y+5 x^{3} y^{2}-7 x^{3} y^{2}+x^{4} y-6 x^{3} y^{2}+10 x y^{4}$
2. Evaluate $\frac{x^{4}-y^{4}}{x^{2}+y^{2}}$ if $\boldsymbol{x}=-4$ and 1 .
3. Find the sum of $\boldsymbol{P}_{\mathbf{1}}=\mathbf{9} \boldsymbol{x}^{\mathbf{6}} \boldsymbol{y}^{\mathbf{3}}-\mathbf{7} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{\mathbf{5}}-\mathbf{2} \boldsymbol{x} \boldsymbol{y}^{\mathbf{6}}$ and

$$
P_{2}=9 x^{6} y^{3}+12 x^{3} y^{5}+15 x y^{6}
$$

4. Subtract $\boldsymbol{P}_{2}=\mathbf{9} \boldsymbol{x}^{6} \boldsymbol{y}^{3}+12 \boldsymbol{x}^{3} \boldsymbol{y}^{5}+15 \boldsymbol{x} \boldsymbol{y}^{6}$ from

$$
P_{1}=9 x^{6} y^{3}-7 x^{3} y^{5}-2 x y^{6}
$$

5. Find the product of $\boldsymbol{P}_{\mathbf{3}}=\mathbf{9} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{\mathbf{5}}+\mathbf{6} \boldsymbol{x} \boldsymbol{y}^{\mathbf{2}}$ and

$$
P_{4}=9 x^{3} y^{5}-6 x y^{2}
$$

6. A can of soda is a right circular cylinder with top and bottom circles of radius $\boldsymbol{r}$, and cylinder height $\boldsymbol{h}$. The volume is $\boldsymbol{V}=\boldsymbol{\pi} \boldsymbol{r}^{\mathbf{2}} \boldsymbol{h}$. If the volume is $36 \mathrm{~cm}^{\mathbf{3}}$ and the radius is 1.5 cm , find the height of the can. Use $\boldsymbol{\pi} \approx \mathbf{3 . 1 4} \mathrm{cm}$. Round to one decimal.

## STOP!

1. Simplify $3 x^{4} y+5 x^{3} y^{2}-7 x^{3} y^{2}+x^{4} y-6 x^{3} y^{2}+10 x y^{4}$

$$
\begin{aligned}
& \text { Solution: } \\
& \qquad 3 x^{4} y+5 x^{3} y^{2}-7 x^{3} y^{2}+x^{4} y-6 x^{3} y^{2}+10 x y^{4} \\
& =3 x^{4} y+x^{4} y+5 x^{3} y^{2}-7 x^{3} y^{2}-6 x^{3} y^{2}+10 x y^{4} \\
& =4 x^{4} y-2 x^{3} y^{2}+10 x y^{4}
\end{aligned}
$$

2. Evaluate $\frac{\boldsymbol{x}^{4}-\boldsymbol{y}^{4}}{\boldsymbol{x}^{2}+\boldsymbol{y}^{2}}$ if $\boldsymbol{x}=-4$ and 1 .

## Solution:

$$
\begin{aligned}
\frac{x^{4}-y^{4}}{x^{2}+y^{2}} & =\frac{(-4)^{4}-(1)^{4}}{(-4)^{2}+(1)^{2}} \\
& =\frac{256-1}{16+1} \\
& =\frac{255}{17} \\
& =15
\end{aligned}
$$

3. Find the sum of $\boldsymbol{P}_{\mathbf{1}}=\mathbf{9} \boldsymbol{x}^{6} \boldsymbol{y}^{\mathbf{3}}-\mathbf{7} \boldsymbol{x}^{3} \boldsymbol{y}^{5}-\mathbf{2} \boldsymbol{x} \boldsymbol{y}^{6}$ and $\boldsymbol{P}_{\mathbf{2}}=\mathbf{9} \boldsymbol{x}^{6} \boldsymbol{y}^{3}+12 \boldsymbol{x}^{3} \boldsymbol{y}^{5}+\mathbf{1 5 x} \boldsymbol{y}^{6}$

Solution:

$$
\begin{aligned}
& P_{1}+P_{2} \\
= & 9 x^{6} y^{3}-7 x^{3} y^{5}-2 x y^{6}+9 x^{6} y^{3}+12 x^{3} y^{5}+15 x y^{6} \\
= & 9 x^{6} y^{3}+9 x^{6} y^{3}-7 x^{3} y^{5}+12 x^{3} y^{5}-2 x y^{6}+15 x y^{6} \\
= & 18 x^{6} y^{3}+5 x^{3} y^{5}+13 x y^{6}
\end{aligned}
$$

4. Subtract $P_{2}=9 x^{6} y^{3}+12 x^{3} y^{5}+15 x y^{6}$ from $P_{1}=9 x^{6} y^{3}-7 x^{3} y^{5}-2 x y^{6}$

## Solution:

$$
\begin{aligned}
& P_{1}-P_{2} \\
= & 9 x^{6} y^{3}-7 x^{3} y^{5}-2 x y^{6}-\left(9 x^{6} y^{3}+12 x^{3} y^{5}+15 x y^{6}\right) \\
= & 9 x^{6} y^{3}-7 x^{3} y^{5}-2 x y^{6}-9 x^{6} y^{3}-12 x^{3} y^{5}-15 x y^{6} \\
= & 9 x^{6} y^{3}-9 x^{6} y^{3}-7 x^{3} y^{5}-12 x^{3} y^{5}-2 x y^{6}-15 x y^{6} \\
= & -19 x^{3} y^{5}-17 x y^{6}
\end{aligned}
$$

5. Find the product of $\boldsymbol{P}_{\mathbf{3}}=\mathbf{9} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{\mathbf{5}}+\mathbf{6} \boldsymbol{x} \boldsymbol{y}^{\mathbf{2}}$ and $\boldsymbol{P}_{\mathbf{4}}=\mathbf{9} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}^{\mathbf{5}}-\mathbf{6} \boldsymbol{x} \boldsymbol{y}^{\mathbf{2}}$

## Solution:

$$
\begin{aligned}
& P_{3} P_{4} \\
= & \left(9 x^{3} y^{5}+6 x y^{2}\right)\left(9 x^{3} y^{5}-6 x y^{2}\right) \\
= & \left(9 x^{3} y^{5}\right)^{2}-\left(6 x y^{2}\right)^{2} \\
= & 81 x^{6} y^{10}-36 x^{2} y^{4}
\end{aligned}
$$

6. A can of soda is a right circular cylinder with top and bottom circles of radius $\boldsymbol{r}$, and cylinder height $\boldsymbol{h}$. The volume is $\boldsymbol{V}=\boldsymbol{\pi} \boldsymbol{r}^{\mathbf{2}} \boldsymbol{h}$. If the volume is $36 \mathrm{~cm}^{\mathbf{3}}$ and the radius is 1.5 cm , find the height of the can. Use $\boldsymbol{\pi} \approx \mathbf{3 . 1 4} \mathrm{cm}$. Round to one decimal.
Solution:

$$
\begin{aligned}
V & =\pi r^{2} h \\
36 & =(3.14)(1.5)^{2} h \\
(3.14)(1.5)^{2} h & =36
\end{aligned}
$$

$$
\begin{aligned}
h & =\frac{36}{(3.14)(1.5)^{2}} \\
h & =5.1 \mathrm{~cm}
\end{aligned}
$$

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## Chapter 34

## Division of Polynomials

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### 34.1 Youtube

https://www.youtube.com/playlist?list=PL722500CD126FB5FF

### 34.2 Division by a monomial

$\frac{12 x^{6} y^{4}}{2 x^{2} y}-\frac{8 x^{4} y^{3}}{2 x^{2} y}+\frac{6 x^{2} y}{2 x^{2} y}=\frac{12 x^{6} y^{4}-8 x^{4} y^{3}+6 x^{2} y}{2 x^{2} y}$
because the denominators are equal.
Conversely

$$
\begin{aligned}
\frac{12 x^{6} y^{4}-8 x^{4} y^{3}+6 x^{2} y}{2 x^{2} y} & =\frac{12 x^{6} y^{4}}{2 x^{2} y}-\frac{8 x^{4} y^{3}}{2 x^{2} y}+\frac{6 x^{2} y}{2 x^{2} y} \\
& =6 x^{4} y^{3}-4 x^{2} y^{2}+3
\end{aligned}
$$

This last decomposition illustrates division of a polynomial by a monomial.

### 34.3 Division by a polynomial

Example 1:

Divide $x^{3}-6 x^{2}+7 x-2$ by $x-1$.

## Solution:

$$
\begin{aligned}
& x^{2} \\
& x - 1 \longdiv { x ^ { 3 } - 6 x ^ { 2 } + 7 x - 2 } \\
& -x^{3}+x^{2} \quad \text { 2. } x^{2}(x-1)=x^{3}-x^{2}\left(\operatorname{add}-x^{3}+x^{2}\right) \\
& -\boldsymbol{5} \boldsymbol{x}^{\mathbf{2}}+\mathbf{7} \boldsymbol{x} \quad \text { Step 3. Take down next term } \boldsymbol{7} \boldsymbol{x} \\
& \boldsymbol{x}^{2}-5 x \quad \text { Step 1.? multiply } \boldsymbol{x} \text { by to get }-5 \boldsymbol{x}^{2} \\
& x - 1 \longdiv { x ^ { 3 } - 6 x ^ { 2 } + 7 x - 2 } \\
& \begin{array}{r}
-x^{3}+x^{2} \\
-5 x^{2}+7 x
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& \boldsymbol{x}^{2}-5 x+2 \quad \text { Step 1.? multiply } \boldsymbol{x} \text { by to get } \mathbf{2 x} \text { ? } \\
& x - 1 \longdiv { x ^ { 3 } - 6 x ^ { 2 } + 7 x } - 2 \\
& \begin{array}{r}
-x^{3}+x^{2} \\
-5 x^{2}+7 x
\end{array} \\
& 5 x^{2}-5 x \\
& 2 x-2 \\
& -2 x+2 \cdot 2(x-1)=2 x-2(\operatorname{add}-2 x+2) \\
& 0
\end{aligned}
$$

Thus the quotient of $\frac{x^{3}-6 x^{2}+7 x-2}{x-1}$ is $x^{2}-5 x+2$.
If there were a remainder $\boldsymbol{R}$, the above quotient would be
$x^{2}-5 x+2+\frac{R}{x-1}$
What would you expect the product $(x-1)\left(x^{2}-5 x+2\right)$ to yield?

The procedure of long division above parallels that of long division for numbers. To illustrate divide 28,382 by 23


## 12

Step 1. What do you multiply 23 by to get 53 ?
23

| 2 8 3 8 |  |  |
| ---: | :--- | :--- | :--- | :--- |
| -2 | 3 |  |


| -4 | 6 | 2. Subt. $2(23)=46$ (change signs and add -46$)$ |
| ---: | :--- | :--- | :--- |
|  | $\mathbf{7}$ 8$\quad$Step 3. Take down the next digit 8 |  |

23

|  | 1 | 2 | 3 | Step 1. What do you multiply 23 by to get $\mathbf{7 8}$ ? |
| :---: | :---: | :---: | :---: | :---: |
| 2 | 8 | 3 | 8 | 2 |
| -2 | 3 |  |  |  |
|  | 5 | 3 |  |  |
|  | -4 | 6 |  |  |
|  |  | 7 | 8 |  |
|  |  | -6 | 9 | Step 2. Subt. $\mathbf{3 ( 2 3 )}=\mathbf{6 9}$ (change signs; add $\mathbf{- 6 3}$ ) |
|  |  |  | 9 | 2 Step 3. Take down the next term 2 |



Before carrying out long division it is important to ensure that the dividend (the polynomial divided into) has all the terms present starting with the highest degree and that the powers of the terms are descending.

To divide $\mathbf{7 x}+\boldsymbol{x}^{\mathbf{3}}+\mathbf{2}-\mathbf{6} \boldsymbol{x}^{2}$ by a polynomial divisor, rearrange
the dividend as $\boldsymbol{x}^{3}-\mathbf{6} \boldsymbol{x}^{2}+\mathbf{7} \boldsymbol{x}+\mathbf{2}$.
To divide $\boldsymbol{x}^{\mathbf{3}}-\mathbf{8}$ by $\boldsymbol{x}-\mathbf{2}$ add $\mathbf{0} \boldsymbol{x}^{\mathbf{2}}+\mathbf{0} \boldsymbol{x}$ to have all the powers present.

Example 2:
Divide $\boldsymbol{x}^{\mathbf{3}}-\mathbf{8}$ by $\boldsymbol{x}-\mathbf{2}$.

## Solution:

$$
\begin{aligned}
& -x^{3}+2 x^{2} \quad 2 . x^{2}(x-2)=x^{3}-2 x^{2}\left(\mathrm{add}-x^{3}+2 x^{2}\right) \\
& \text { Step 3. Take down the next term } \mathbf{0 x} \\
& \begin{array}{l}
\quad \boldsymbol{x}^{2}+2 \boldsymbol{x} \quad \text { Step 1. multiply } x \text { by } ? \text { to get } 2 x^{2} ? ~
\end{array} \\
& \frac{-x^{3}+2 x^{2}}{2 x^{2}+0 x} \\
& \begin{array}{r}
-2 \boldsymbol{x}^{2}+\mathbf{4 x} \quad 2 . \mathbf{x}(\boldsymbol{x}-\mathbf{2})=\mathbf{2} \boldsymbol{x}^{\mathbf{2}-\mathbf{4 x}\left(\text { add }-\mathbf{2} \boldsymbol{x}^{2}+\mathbf{4 x}\right.} \begin{array}{r}
\text { 4x-8}
\end{array} \text { Step 3. Take down next term }-\mathbf{8}
\end{array} \\
& \begin{array}{l}
x-2 x^{2}+2 x+4 \\
x^{3}+0 x^{2}+0 x-8
\end{array} \\
& \frac{-x^{3}+2 x^{2}}{2 x^{2}+0 x} \\
& \frac{-2 x^{2}+4 x}{4 x-8} \\
& \frac{-4 x+8}{0} 2.4(x-2)=4 x-8(\operatorname{add}-4 x+8)
\end{aligned}
$$

Thus the quotient of $\frac{x^{3}-8}{x-2}$ is $\boldsymbol{x}^{2}+\mathbf{x}+\mathbf{4}$.
What should expect the product $(x-2)\left(x^{2}+4 x+8\right)$ yield?

### 34.4 Exercise 34

1. Divide $\frac{15 a^{7} b^{5}-35 a^{5} b^{3}+5 a^{3} b}{5 a^{3} b}$
2. Divide $8 x^{3}+2 x^{2}-x+22$ by $2 x+3$.
3. Divide $8 x^{3}-27$ by $2 x-3$.

STOP!

1. Divide $\frac{15 a^{7} b^{5}-35 a^{5} b^{3}+5 a^{3} b}{5 a^{3} b}=\frac{15 a^{7} b^{5}}{5 a^{3} b}-\frac{35 a^{5} b^{3}}{5 a^{3} b}+\frac{5 a^{3} b}{5 a^{3} b}$

$$
=3 a^{4} b^{4}-7 a^{2} b^{2}+1
$$

2. Divide $8 x^{3}+\mathbf{2} \boldsymbol{x}^{2}-x+22$ by $2 x+3$.

## Solution:

Thus the quotient of $\frac{8 x^{3}+2 x^{2}-x+122}{2 x+3}$ is $4 x^{2}-5 x+7+\frac{1}{2 x+3}$.
What would you expect the product $(2 x+3)\left(4 x^{2}-5 x+7+\frac{1}{2 x+3}\right)$ to yield?
3. Divide $8 \boldsymbol{x}^{\mathbf{3}}-\mathbf{2 7}$ by $\mathbf{2 x}-\mathbf{3}$.

Solution:


Step 3. Take down the next term $\mathbf{0} \boldsymbol{x}$

$$
\begin{aligned}
& 4 x^{2} \quad \text { Step 1. multiply } 2 x \text { by ? for } 8 x^{3} \\
& 2 x+3 \quad 8 x^{3}+2 x^{2}-x+22 \\
& \begin{array}{rll}
-8 \boldsymbol{x}^{3}-\mathbf{1 2 x} & & \text { 2. } \mathbf{4} \boldsymbol{x}^{2}(\mathbf{2 x + 3})=8 x^{3}+\mathbf{1 2} x^{2}\left(\operatorname{add}-\mathbf{4} x^{3}-\mathbf{1 2} x^{2}\right) \\
-10 \boldsymbol{x}^{2}-x & \text { Step 3. Take down next term }-x
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& 2 x-3 \begin{array}{l}
4 x^{2}+6 x \\
\cline { 2 - 3 }+0 x^{3}+0 x^{2}+0 x-27
\end{array} \\
& \frac{-8 x^{3}+12 x^{2}}{12 x^{2}+0 x} \\
& \begin{array}{r}
-12 x^{2}+18 x \quad 2.6 x(2 x-3)=12 x^{2}-18 x\left(\text { add }-6 x^{2}+18 x\right) \\
18 x-27
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& 4 x^{2}+6 \boldsymbol{x}+\mathbf{9} \quad \text { Step 1. multiply } 2 \boldsymbol{x} \text { by ? to get } 18 x \text { ? } \\
& 2 x - 3 \longdiv { 8 x ^ { 3 } + 0 x ^ { 2 } + 0 x - 2 7 } \\
& \frac{-8 x^{3}+12 x^{2}}{12 x^{2}+0 x} \\
& \begin{array}{r}
-12 x^{2}+18 x \\
18 x-27
\end{array} \\
& \frac{\left.-18 x+{ }_{27} 9(2 x-3)=18 x-27 \Rightarrow-18 x+27\right)}{0}
\end{aligned}
$$

Thus the quotient of $\frac{8 x^{3}-27}{2 x-3}$ is $4 x^{2}+6 x+9$.
What would you expect the product $(2 \boldsymbol{x}-\mathbf{3})\left(\mathbf{4}^{2}+\mathbf{6 x}+\mathbf{9}\right)$ to yield?

## Chapter 35

## Negative Exponents

(C) H. Feiner 2011

### 35.1 Youtube

https://www.youtube.com/playlist?list=PLF41ACBEC924C0887\&feature=view_all

### 35.2 Basics

We remember that $2^{3} \cdot 2^{4}=(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)=2^{3+4}=2^{7}$
In general

$$
a^{m} a^{n}=a^{m+n}
$$

We also remember that
$\left(2^{3}\right)^{4}=(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2)=2^{3 \cdot 4}=2^{12}$
In general

$$
\left(a^{m}\right)^{n}=a^{m n}
$$

$\frac{2^{9}}{2^{4}}=\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{1}=2^{9-4}=2^{5}$
In general

$$
\frac{a^{m}}{a^{n}}=a^{m-n}
$$

Suppose $\boldsymbol{m}<\boldsymbol{n}$.
For example $\frac{2^{3}}{2^{\mathbf{7}}}=\mathbf{2}^{\mathbf{3 - 7}}=\mathbf{2}^{-4}$ and
$\frac{2^{3}}{2^{7}}=\frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{1}{2 \cdot 2 \cdot 2 \cdot 2}=\frac{1}{2^{4}}$
Both $\frac{1}{2^{4}}$ and $\mathbf{2}^{-4}$ came from the same fraction $\frac{2^{3}}{2^{7}}$, so $2^{-4}=\frac{1}{2^{4}}$
In general

$$
a^{-m}=\frac{1}{a^{m}}
$$

Note that $\boldsymbol{a}^{-\boldsymbol{m}}$ and $\frac{\mathbf{1}}{\boldsymbol{a}^{\boldsymbol{m}}}$ are both positive (or negative). Only the sign of the exponents change.
Now suppose $\boldsymbol{m}=\boldsymbol{n}($ and $\boldsymbol{a} \neq \mathbf{0})$.
For example $\frac{2^{3}}{2^{3}}=2^{3-3}=2^{0}$ and
$\frac{2^{3}}{2^{3}}=\frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}=1$
Both
$\frac{2^{3}}{2^{3}}$ and $\mathbf{2}^{0}$ came from the same fraction $\frac{2^{3}}{2^{3}}$, so $2^{0}=1$
In general

$$
a^{0}=1(a \neq 0)
$$

### 35.3 Examples

Example 1:
Simplify $\left(\mathbf{3}^{-2} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}^{-\mathbf{5}}\right)\left(\mathbf{9}^{-\mathbf{3}} \boldsymbol{y}^{-\mathbf{4}}\right)$. Leave no negative exponents.

## Solution:

$$
\begin{aligned}
\left(3^{-2} x^{2} y^{-5}\right)\left(9 x^{-3} y^{-4}\right) & =\left(3^{-2}\right)(9) x^{2} x^{-3} y^{-5} y^{-4} \\
& =\left(3^{-2}\right)\left(3^{2}\right) x^{2-3} y^{-5-4} \\
& =3^{-2+2} x^{-1} y^{-9} \\
& =\frac{3^{0}}{x y^{9}} \\
& =\frac{1}{x y^{9}}
\end{aligned}
$$

Example 2:
Simplify $\frac{\mathbf{3}^{-\mathbf{2}} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}^{-\mathbf{5}}}{\mathbf{9} \boldsymbol{x}^{-\mathbf{3}} \boldsymbol{y}^{-4}}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\frac{3^{-2} x^{2} y^{-5}}{9 x^{-3} y^{-4}} & =\frac{x^{2} x^{3} y^{4}}{(9)\left(3^{2}\right) y^{5}} \\
& =\frac{x^{2+3}}{(9)(9) y} \\
& =\frac{x^{5}}{81 y}
\end{aligned}
$$

Example 3:
Simplify $\left(\frac{\mathbf{3}^{-\mathbf{2}} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}^{-\mathbf{5}}}{\mathbf{9} \boldsymbol{x}^{-\mathbf{3}} \boldsymbol{y}^{-\mathbf{4}}}\right)^{\mathbf{0}}$ and leave no negative exponents.

## Solution:

$\left(\frac{3^{-2} x^{2} y^{-5}}{9 x^{-3} y^{-4}}\right)^{0}=1$
Any non-zero quantity to the power 0 is 1 .
Example 4:
Simplify $\left(\frac{\left(3^{-2} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}^{-5}\right)^{2}}{\mathbf{9} \boldsymbol{x}^{-\mathbf{3}} \boldsymbol{y}^{-4}}\right)^{-\mathbf{1}}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{\left(3^{-2} x^{2} y^{-5}\right)^{2}}{9 x^{-3} y^{-4}}\right)^{-1} & =\left(\frac{\left(3^{-2} x^{2} y^{-5}\right)^{2}}{9 x^{-3} y^{-4}}\right)^{-1} \\
& =\left(\frac{9 x^{-3} y^{-4}}{\left(3^{-2} x^{2} y^{-5}\right)^{2}}\right)^{1} \text { note }\left(\frac{a}{b}\right)^{-1}=\frac{1}{\frac{a}{b}}=1 \cdot \frac{b}{a}=\frac{b}{a} \\
& =\frac{9 x^{-3} y^{-4}}{3^{-2 \cdot 2} x^{2 \cdot 2} y^{-5 \cdot 2}} \\
& =\frac{3^{2} x^{-3} y^{-4}}{3^{-4} x^{4} y^{-10}} \\
& =\frac{\left(3^{2}\right)\left(3^{4}\right) y^{10}}{x^{4} x^{3} y^{4}} \\
& =\frac{3^{2+4} y^{10-4}}{x^{4+3}} \\
& =\frac{3^{6} y^{6}}{x^{7}}
\end{aligned}
$$

Example 5:
Simplify $\frac{\mathbf{3}^{-\mathbf{2}} \mathbf{4}^{-\mathbf{1}}}{\mathbf{5}^{-\mathbf{3}}}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\frac{3^{-2} 4^{-1}}{5^{-3}} & =\frac{5^{3}}{3^{2} 4^{1}} \\
& =\frac{125}{(9)(4)} \\
& =\frac{125}{36}
\end{aligned}
$$

Example 6:
Simplify $\frac{\mathbf{3}^{-\mathbf{2}}-\mathbf{4}^{-\mathbf{1}}}{\mathbf{5}^{-\mathbf{3}}}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\frac{3^{-2}-4^{-1}}{5^{-3}} & =\frac{\frac{1}{3^{2}}-\frac{1}{4}}{\frac{1}{5^{3}}} \\
& =\frac{\frac{4}{\left(3^{2}\right)(4)}-\frac{3^{2}}{(4)\left(3^{2}\right)}}{\frac{1}{5^{3}}} \\
& =\frac{\frac{4}{(9)(4)}-\frac{9}{(4)(9)}}{\frac{1}{125}} \\
& =\frac{\frac{4}{36}-\frac{9}{36}}{\frac{1}{125}} \\
& =\frac{\frac{4-9}{36}}{\frac{1}{125}} \\
& =\frac{-\frac{5}{36}}{\frac{1}{125}} \\
& =\frac{5}{36} \cdot(125)=-\frac{625}{36}
\end{aligned}
$$

### 35.4 Exercise 35

1. Simplify $\left(\mathbf{2}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{4}} \boldsymbol{y}^{-\mathbf{5}}\right)\left(\mathbf{8} \boldsymbol{x}^{-\mathbf{6}} \boldsymbol{y}^{-\mathbf{1}}\right)$ and leave no negative exponents.
2. Simplify $\frac{\mathbf{8 x}^{-\mathbf{6}} \boldsymbol{y}^{-\mathbf{1}}}{\mathbf{2}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{4}} \boldsymbol{y}^{-\mathbf{5}}}$ and leave no negative exponents.
3. Simplify $\left(\frac{\mathbf{6}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{5}} \boldsymbol{y}^{-7}}{\mathbf{9} \boldsymbol{x}^{-\mathbf{1}} \boldsymbol{y}^{-8}}\right)^{\mathbf{0}}$. Leave no negative exponents.
4. Simplify $\left(\frac{\left(4^{-3} a^{2} b^{-7}\right)^{\mathbf{3}}}{16 a^{-5} b^{-6}}\right)^{-\mathbf{1}}$. Leave no negative exponents.
5. Simplify $\frac{\mathbf{2}^{-\mathbf{3}} \boldsymbol{5}^{-\mathbf{2}}}{\mathbf{6}^{-\mathbf{1}}}$ and leave no negative exponents.
6. Simplify $\frac{\mathbf{6}^{-\mathbf{3}}-\mathbf{5}^{-\mathbf{3}}}{\mathbf{3 0}^{-\mathbf{2}}}$ and leave no negative exponents.

## STOP!

1. Simplify $\left(\mathbf{2}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{4}} \boldsymbol{y}^{-\mathbf{5}}\right)\left(8 \boldsymbol{x}^{-\mathbf{6}} \boldsymbol{y}^{-\mathbf{1}}\right)$ and leave no negative exponents.

Solution:

$$
\begin{aligned}
\left(2^{-3} x^{4} y^{-5}\right)\left(8 x^{-6} y^{-1}\right) & =\left(2^{-3} x^{4} y^{-5}\right)\left(2^{3} x^{-6} y^{-1}\right) \\
& =2^{-3} 2^{3} x^{4} x^{-6} y^{-5} y^{-1} \\
& =2^{-3+3} x^{4-6} y^{-5-1} \\
& =2^{0} x^{-2} y^{-6} \\
& =\frac{1}{x^{2} y^{6}}
\end{aligned}
$$

2. Simplify $\frac{\boldsymbol{B}^{-\mathbf{6}} \boldsymbol{y}^{-\mathbf{1}}}{\mathbf{2}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{4}} \boldsymbol{y}^{-\mathbf{5}}}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\frac{8 x^{-6} y^{-1}}{2^{-3} x^{4} y^{-5}} & =\frac{\left(2^{3}\right)\left(2^{3}\right) y^{5}}{x^{4} x^{6} y^{1}} \\
& =\frac{(8)(8) y^{5-1}}{x^{4+6}} \\
& =\frac{64 y^{4}}{x^{10}}
\end{aligned}
$$

3. Simplify $\left(\frac{\mathbf{6}^{-\mathbf{3}} \boldsymbol{x}^{\mathbf{5}} \boldsymbol{y}^{-\mathbf{7}}}{\mathbf{9} \boldsymbol{x}^{-\mathbf{1}} \boldsymbol{y}^{-\mathbf{8}}}\right)^{\mathbf{0}}$ and leave no negative exponents.

## Solution:

$$
\left(\frac{6^{-3} x^{5} y^{-7}}{9 x^{-1} y^{-8}}\right)^{0}=1
$$

Any non-zero quantity to the power 0 is 1 .
4. Simplify $\left(\frac{\left(4^{-3} \boldsymbol{a}^{2} b^{-7}\right)^{3}}{16 a^{-5} b^{-6}}\right)^{-1}$ and leave no negative exponents.

## Solution:

$$
\begin{aligned}
\left(\frac{\left(4^{-3} a^{2} b^{-7}\right)^{3}}{16 a^{-5} b^{-6}}\right)^{-1} & =\left(\frac{\left(4^{-3} a^{2} b^{-7}\right)^{3}}{4^{2} a^{-5} b^{-6}}\right)^{-1} \\
& =\frac{4^{2} a^{-5} b^{-6}}{\left(4^{-3} a^{2} b^{-7}\right)^{3}} \\
& =\frac{4^{2} a^{-5} b^{-6}}{4^{-3(3)} a^{2(3)} b^{-7(3)}} \\
& =\frac{4^{2} a^{-5} b^{-6}}{4^{-9} a^{6} b^{-21}} \\
& =\frac{4^{2-(-9)} a^{-5-6} b^{-6-(-21)}}{1} \\
& =\frac{4^{11} a^{-11} b^{15}}{1} \\
& =\frac{4^{11} b^{15}}{a^{11}}
\end{aligned}
$$

5. Simplify $\frac{\mathbf{2}^{-\mathbf{3}} \mathbf{5}^{-\mathbf{2}}}{\mathbf{6}^{-\mathbf{1}}}$ and leave no negative exponents.

Solution:

$$
\begin{aligned}
\frac{2^{-3} 5^{-2}}{6^{-1}} & =\frac{6^{1}}{\left(2^{3}\right)\left(5^{2}\right)} \\
& =\frac{6}{(8)(25)} \\
& =\frac{3}{(4)(25)} \\
& =\frac{3}{100}
\end{aligned}
$$

6. Simplify $\frac{\mathbf{6}^{-\mathbf{3}}-\mathbf{5}^{-\mathbf{3}}}{\mathbf{3 0}^{-\mathbf{2}}}$ and leave no negative exponents.

Solution:

$$
\begin{aligned}
& \frac{6^{-3}-5^{-3}}{30^{-2}}=\frac{\frac{1}{6^{3}}-\frac{1}{5^{3}}}{\frac{1}{30^{2}}} \\
& =\frac{\frac{5^{3}}{\left(6^{3}\right)\left(5^{3}\right)}-\frac{6^{3}}{\left(5^{3}\right)\left(6^{3}\right)}}{\frac{1}{30^{2}}} \\
& =\frac{\frac{125}{\left(6^{3}\right)\left(5^{3}\right)}-\frac{6^{3}}{\left(5^{3}\right)\left(6^{3}\right)}}{\frac{1}{30^{2}}} \\
& =\frac{\frac{125-216}{\left(6^{3}\right)\left(5^{3}\right)}}{\frac{1}{30^{2}}} \\
& =\frac{-91}{\left(6^{3}\right)\left(5^{3}\right)}\left(30^{2}\right) \\
& =\frac{-91}{\left(6^{3}\right)\left(5^{3}\right)}(5 \cdot 6)^{2} \\
& =\frac{(-91)\left(5^{2}\right)\left(6^{2}\right)}{\left(6^{3}\right)\left(5^{3}\right)} \\
& =\frac{-91}{(6)(5)} \\
& =\frac{-91}{30}
\end{aligned}
$$

## Chapter 36

## Scientific Notation

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### 36.1 Youtube

https://www.youtube.com/playlist?list=PLB93357D874AB8A4D\&feature=view_all

### 36.2 Basics

I square 123456789123456789 on my scientific calculator and read the result to be 1.524157818 .
What does that mean?
The result is supposed to be
$1.5241578 \times 10^{18}=1,524,157,800,000,000,000$.
The result is an approximation. The display in my calculator window is limited. The zeros above mask the actual digits.

The number $1.5241578 \times 18$ is written in scientific notation.

A number is written in scientific notation if it is in the form

$$
N \times 10^{m}
$$

where $1 \leq N<10$ and $m$ is an integer.

The mass of the moon is $73,494,123,567,890,123,456,789 \mathrm{~kg}$.
Memorize that number.
What? Have you lost your mind?
Actually all but the first four digits are fictitious to make a point (see en.wikipedia.org/wiki/Earth\#Moon)
Remembering $73,490,000,000,000,000,000,000$ is good enough and a lot simpler to remember.
Written in scientific notation, this number is $\mathbf{7 . 3 4 9} \times \mathbf{1 0}^{\mathbf{2 2}}$.
Planck's constant in physics is $\mathbf{6 . 2 6 0 6 8 9 6} \times \mathbf{1 0}^{\mathbf{- 3 4}}$ Joules-second.
What is this number in decimal notation? It is
0.00000000000000000000000000000000062606896.

Remember that $\mathbf{1 0}^{-\mathbf{3 4}}=\frac{\mathbf{1}}{\mathbf{1 0}^{\mathbf{3 4}}}$.

### 36.3 Examples

Example 1:
Write 56, 708 in scientific notation.

## Solution:

$56,708=5.6708 \times 10^{4}$
Example 2:
Write 0.0056708 in scientific notation.

## Solution:

$0.0056708=5.6708 \times 10^{-3}$
Example 3:
Find the sum of $\mathbf{1 . 2} \times \mathbf{1 0}^{\mathbf{3}}$ and $\mathbf{3 . 4} \times \mathbf{1 0}^{\mathbf{2}}$ in scientific notation.

## Solution:

$$
\begin{aligned}
\left(1.2 \times 10^{3}\right)+\left(3.4 \times 10^{2}\right) & =1,200+340 \\
& =1,540 \\
& =1.54 \times 10^{3}
\end{aligned}
$$

## Example 4:

Find the product of $\mathbf{4 . 5} \times \mathbf{1 0}^{\mathbf{3}}$ and $\mathbf{3 . 1} \times \mathbf{1 0}^{\mathbf{2}}$ without first converting to decimal notation. Give the product in scientific notation.

## Solution:

$$
\begin{aligned}
\left(4.5 \times 10^{3}\right)\left(3.1 \times 10^{2}\right) & =(4.5)(3.1)\left(10^{3}\right)\left(10^{2}\right) \\
& =(13.95) \times 10^{3+2} \\
& =(1.395 \cdot 10) \times 10^{5} \\
& =1.395 \times 10^{6}
\end{aligned}
$$

Example 5:
Find the quotient of $\mathbf{4 . 3} \times \mathbf{1 0}^{\mathbf{6}}$ and $\mathbf{8 . 6} \times \mathbf{1 0}^{\mathbf{2}}$ without first converting to decimal notation. Give the quotient in scientific notation.

## Solution:

$$
\begin{aligned}
\frac{4.3 \times 10^{6}}{8.6 \times 10^{2}} & =\frac{1 \times 10^{6}}{2 \times 10^{2}} \\
& =0.5 \times 10^{6-2} \\
& =0.5 \times 10^{4} \\
& =(0.5)(10) \times 10^{3} \\
& =5 \times 10^{3}
\end{aligned}
$$

### 36.4 Exercise 36

1. Write 259,000 in scientific notation.
2. Write $\mathbf{0 . 0 0 0} 000 \mathbf{8}$ in scientific notation.
3. Find the sum of $\mathbf{4 . 5} \times \mathbf{1 0}^{\mathbf{3}}$ and $\mathbf{7 . 4} \times \mathbf{1 0}^{\mathbf{4}}$ in scientific notation.
4. Find the product of $\mathbf{7 . 3} \times \mathbf{1 0}^{\mathbf{4}}$ and $\mathbf{3 . 2} \times \mathbf{1 0}^{\mathbf{5}}$ without first converting to decimal notation. Give the product in scientific notation.
5. Find the quotient of $\mathbf{2 . 7} \times \mathbf{1 0}^{\mathbf{7}}$ and $\mathbf{5 . 4} \times \mathbf{1 0}$ without first converting to decimal notation. Give the quotient in scientific notation.

## STOP!

1. Write $\mathbf{2 5 9}, 000$ in scientific notation.

## Solution:

$259,000=2.59 \times 10^{5}$
2. Write $\mathbf{0 . 0 0 0} 0008$ in scientific notation.

## Solution:

$0.0000008=8 \times 10^{-7}$
3. Find the sum of $\mathbf{4 . 5} \times \mathbf{1 0}^{\mathbf{3}}$ and $\mathbf{7 . 4} \times \mathbf{1 0}^{\mathbf{4}}$ in scientific notation.

## Solution:

$$
\begin{aligned}
\left(4.5 \times 10^{3}\right)+\left(7.4 \times 10^{4}\right) & =4,500+74,000 \\
& =78,500 \\
& =7.85 \times 10^{4}
\end{aligned}
$$

4. Find the product of $\mathbf{7 . 3} \times \mathbf{1 0}^{\mathbf{4}}$ and $\mathbf{3 . 2} \times \mathbf{1 0}^{\mathbf{5}}$ without first converting to decimal notation. Give the product in scientific notation.

## Solution:

$$
\begin{aligned}
\left(7.3 \times 10^{4}\right)\left(3.2 \times 10^{5}\right) & =(7.3)(3.2)\left(10^{4}\right)\left(10^{5}\right) \\
& =(23.36) \times 10^{4+5} \\
& =(23.36) \times 10^{9} \\
& =(2.336 \cdot 10) \times 10^{9} \\
& =2.336 \times 10^{10}
\end{aligned}
$$

5. Find the quotient of $\mathbf{2 . 7} \times \mathbf{1 0}^{\mathbf{7}}$ and $\mathbf{5 . 4} \times \mathbf{1 0}$ without first converting to decimal notation. Give the quotient in scientific notation.

## Solution:

$$
\begin{aligned}
\frac{2.7 \times 10^{7}}{5.4 \times 10} & =\frac{1 \times 10^{7}}{2 \times 10} \\
& =0.5 \times 10^{7-1} \\
& =0.5 \times 10^{6} \\
& =(0.5)(10) \times 10^{5} \\
& =5 \times 10^{5}
\end{aligned}
$$

## Chapter 37

## Factoring the Greatest Common Factor (GCF)

(c) H. Feiner 2011

### 37.1 Youtube

https://www.youtube.com/playlist?list=PL2483E676747CF66F\&feature=view_all

### 37.2 Basics

A factor is any of two or more mathematical quantities which form a product when multiplied together. For example
in the equation $5 \times 4=204$ and 5 are factors.
In $\mathbf{5 x} 5$ and $\boldsymbol{x}$ are factors.
In $5 x(3-x) 5, x$, and $3-x$ are factors.
A term is any of two or more mathematical quantities which form a sum (or difference) when added together.

Two or more terms can have common factors, like $\mathbf{1 8} \boldsymbol{x}^{\mathbf{2}} \boldsymbol{y}$ and $\mathbf{2 4 x} \boldsymbol{y}^{\mathbf{3}}$ have $\mathbf{2 x}$ in common. $\mathbf{2}$ and $\boldsymbol{x}$ are common factors (among others.) Let us do a primary factorization of the two terms and omit the use of exponents. Write common factors in the same column. Do no mix different factors in the same column.

$$
\begin{array}{lllllllllllll}
18 x^{2} y & = & 2 & & & 3 & 3 & x & x & y & & \\
24 x y^{3} & = & 2 & 2 & 2 & 3 & & x & & y & y & y & \\
\hline 24 x y^{3} & = & 2 & & 3 & x & y & & =6 \boldsymbol{y} \boldsymbol{y} \text { (greatest common factor). }
\end{array}
$$

You were possibly taught other methods of finding the greatest common factor (GCF). I suggest that you learn the way I am showing you now because you will reap benefits when we add/subtract fractions in conjunction with the least common multiple (LCM).

### 37.3 Examples

Example 1:
Find the GCF of $\mathbf{8 4 x} \boldsymbol{x} \boldsymbol{y}^{\mathbf{3}} \boldsymbol{z}^{2}, \mathbf{1 4 0} \boldsymbol{x}^{2} \boldsymbol{y}^{2} z^{2}$, and $\mathbf{5 6 x} \boldsymbol{x} \boldsymbol{y}^{2} \boldsymbol{z}^{2}$.

## Solution:

$$
\begin{array}{rllllllllllllll}
84 x y^{3} z^{2} & = & 2 & 2 & & 3 & & 7 & x & & y & y & y & z & z \\
140 x^{2} y^{2} z^{2} & = & 2 & 2 & & & 5 & 7 & x & x & y & y & & z & z \\
56 x y^{2} z^{2} & = & 2 & 2 & 2 & & 7 & x & & y & y & & z & z \\
\hline & = & 2 & & & 7 & x & & y & y & & z & z
\end{array}
$$

The greatest common factor is $(\mathbf{2})(\mathbf{2})(\mathbf{7}) \boldsymbol{x} \boldsymbol{y}^{\mathbf{2}} \boldsymbol{z}^{2}=\mathbf{2 8} \boldsymbol{x} \boldsymbol{y}^{\mathbf{2}} \boldsymbol{z}^{\mathbf{2}}$
Example 2:
Find the GCF of $4 x(y-5)^{3}, 8 x^{2}(y-5)^{2}(z+3)^{2}$, and $16 x(y-5)^{3}(z+3)^{2}$.

Solution:

$$
\begin{array}{rlllllllll}
4 x(y-5)^{3} & =2 & 2 & & & x & (y-5) & (y-5) & (y-5) \\
8 x^{2}(y-5)^{2}(z+3)^{2} & =2 & 2 & 2 & & x & x & (y-5) & (y-5) & \\
16 x(y-5)^{3}(z+3)^{2} & =2 & 2 & 2 & 2 & x & (y-5) & (y-5) & (y-5)(z+3)(z+3) \\
& =2 & 2 & & x & (y-5) & (y-5) &
\end{array}
$$

The greatest common factor is $\mathbf{4 x}(\boldsymbol{y}-5)^{2}$
We are interested in rewriting a sum or difference of terms as a product of factors. This is known as factoring.

The very first step in the process is factoring GCF (if possible).
This is so important that studenst must get conditioned like Pavlov's dog. As soon as students hears the word "factor", GCF must come to mind immediately.

If you fail to factor the GCF, you may either be prevented from further factoring (if that is possible after extracting the GCF) or you will encounter numbers that are larger than they need to be.
Example 3:
Factor the GCF of $\mathbf{3 6} \boldsymbol{x}^{\mathbf{3}}-\mathbf{1 2 0} \boldsymbol{x}^{\mathbf{2}}+\mathbf{1 0 8 x}$

## Solution:


$12 \boldsymbol{x}$ is the greatest common factor.

$$
36 x^{3}-120 x^{2}+108 x=12 x\left(3 x^{2}-10 x+9\right)
$$

Note that when the numbers are small you can get your result without the seemingly complicated steps shown. You are really going through these steps mentally without writing them down.

Use the distributive property to verify that you have the correct answer.

$$
12 x\left(3 x^{2}-10 x+9\right)=36 x^{3}-120 x^{2}+108 x
$$

Example 4
Factor $(x-7)(5 x+3)-(x-7)(x+9)$

## Solution:

$$
\begin{aligned}
(x-7)(5 x+3)-(x-7)(x+9) & =(x-7)[(5 x+3)-(x+9)] \\
& =(x-7)(5 x+3-x-9) \\
& =(x-7)(4 x-6)
\end{aligned}
$$

The next example shows factoring by grouping.
Example 5
Factor $2 p x-p^{2}-3 p+6 x$

## Solution:

$$
\begin{aligned}
20 p x-10 p^{2}-30 p+60 x & =10\left(2 p x-p^{2}-3 p+6 x\right) & & \text { Factor GCF } 10 \\
10\left(2 p x-p^{2}-3 p+6 x\right) & =10[p(2 x-p)-3(p-2 x)] & & \text { pair terms } \\
& =10[p(2 x-p)+3(2 x-p)] & & \text { factor }-1 \\
& =10(2 x-p)(p+3) & &
\end{aligned}
$$

Instead of pairing $\mathbf{2 p \boldsymbol { p }}$ and $\boldsymbol{p}^{\mathbf{2}}$, you have paired $\mathbf{2 p \boldsymbol { x }}$ and $\mathbf{6 \boldsymbol { x }}$.

$$
\begin{array}{rlrl}
10\left(2 p x+6 x-p^{2}-3 p\right) & =10[2 x(p+3)-p(p+3)] & \text { pair terms and } \\
& =10(p+3)(2 x-p)] \quad \text { factor GCF from each pair } \\
& =10 c t o r ~ a n o t h e r ~ G C F ~
\end{array}
$$

### 37.4 Exercise 37

1. Find the GCF of $\mathbf{9 2 4} \boldsymbol{x}^{\mathbf{7}}, \mathbf{4 2 0} \boldsymbol{x}^{\mathbf{5}}$, and $\mathbf{8 4} \boldsymbol{x}^{\mathbf{6}}$.
2. Find the GCF of $5 a b^{2}(c-5)^{4}, 10 a^{2} b^{2}(c-5)^{2}(d+6)^{2}$, and $20 a b^{2}(c-5)^{3}(d+6)$.
3. Factor the GCF of $\mathbf{8 4} \boldsymbol{x}^{\mathbf{3}}-\mathbf{1 2 6} \boldsymbol{x}^{4}+\mathbf{2 1 0} \boldsymbol{x}^{\mathbf{5}}$
4. Factor $(a+2)(6 a+3)-(a+2)(a+7)+(a+2)$
5. Factor $5 a^{2} x-10 a^{2}-35 x+70$

## STOP!

1. Find the GCF of $\mathbf{9 2 4} \boldsymbol{x}^{\mathbf{7}}, \mathbf{4 2 0} \boldsymbol{x}^{\mathbf{5}}$, and $\mathbf{8 4 \boldsymbol { x } ^ { 6 }}$.

Solution:

| $924 x^{7}$ | $=$ | 2 | 2 | 3 |  | 7 | 11 | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $420 x^{5}$ | $=$ | 2 | 2 | 3 | 5 | 7 |  | $x$ | $x$ | $x$ | $x$ | $x$ |  |  |
| $84 x^{6}$ | $=$ | 2 | 2 | 3 |  | 7 |  | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ |  |
|  | $=$ | 2 | 2 | 3 |  | 7 |  | $x$ | $x$ | $x$ | $x$ | $x$ |  |  |

The greatest common factor is $(2)(2)(3)(7) x^{5}=84 x^{5}$
2. Find the GCF of $5 a b^{2}(c-5)^{4}, 10 a^{2} b^{2}(c-5)^{2}(d+6)^{2}$, and $20 a b^{2}(c-5)^{3}(d+6)$.

## Solution:

$$
5 a b^{2}(c-5)^{4}=\quad 5 \quad a \quad b \quad b \quad(c-5) \quad(c-5)(c-5)(c-5)
$$

$10 a^{2} b^{2}(c-5)^{2}(d+6)^{2}=2 \quad 5 \quad a \quad a b \quad b \quad(c-5) \quad(c-5) \quad(d+6)(d+6)$

| $20 a b^{2}(c-5)^{3}(d+6)$ | 2 | 2 | 5 | $a$ | $b$ | $b$ | $(c-5)$ | $(c-5)(c-5)$ | $(d+6)$ |
| ---: | :--- | ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | 5 | $a$ | $b$ | $b$ | $(c-5)$ | $(c-5)$ |  |  |  |

The greatest common factor is $\mathbf{5 a} \boldsymbol{b}^{\mathbf{2}}(\boldsymbol{c}-\mathbf{5})^{\mathbf{2}}$
3. Factor the GCF of $\mathbf{8 4} \boldsymbol{x}^{\mathbf{3}}-\mathbf{1 2 6} \boldsymbol{x}^{4}+\mathbf{2 1 0} \boldsymbol{x}^{\mathbf{5}}$

## Solution:

$$
\left.\begin{array}{rlllllllllll}
84 x^{3} & = & 2 & 2 & 3 & & & 7 & x & x & x & \\
126 x^{4} & = & 2 & & 3 & 3 & & 7 & x & x & x & x
\end{array}\right]
$$

$42 x^{3}$ is the greatest common factor.
4. Factor $(a+2)(6 a+3)-(a+2)(a+7)+(a+2)$

## Solution:

$$
\begin{aligned}
& (a+2)(6 a+3)-(a+2)(a+7)+(a+2) \\
= & (a+2)[(6 a+3)-(a+7)+1] \\
= & (a+2)(6 a+3-a-7+1) \\
= & (a+2)(5 a-3)
\end{aligned}
$$

5. Factor $5 a^{2} x-10 a^{2}-35 x+70$

Solution:

$$
\begin{aligned}
5 a^{2} x-10 a^{2}-35 x+70 & =5\left(a^{2} x-2 a^{2}-7 x+14\right) & & \text { Factor GCF } 5 \\
& =5\left[a^{2}(x-2)-7(x-2)\right] & & \text { pair terms } \\
& =5(x-2)\left(a^{2}-7\right) & & \text { factor another GCF }
\end{aligned}
$$

## Chapter 38

## Factoring binomials

(C) H. Feiner 2011

### 38.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqRmCYT16AwTz-yPsKZUC6yd\&feature=view_all

### 38.2 Basics

Recall that a polynomial is a sum or difference of terms (could be in more than one variable) in the form $\boldsymbol{a} \boldsymbol{x}^{\boldsymbol{n}}$ where $\boldsymbol{a}$ is a coefficient and $\boldsymbol{n}$ is a whole number. If a polynomial consists of exactly two terms, then it is a binomial. The prefix "bi" refers to two, as in bicycle, bilingual, biweekly, bifocal, bipartisan, ...

We consider factoring binomials of the form
(1) $\boldsymbol{a}^{2}+\boldsymbol{b}^{\mathbf{2}}$ (a sum of squares)
(2) $\boldsymbol{a}^{\mathbf{2}}-\boldsymbol{b}^{\mathbf{2}}$ (a difference of squares)
(3) $\boldsymbol{a}^{\mathbf{3}}+\boldsymbol{b}^{\mathbf{3}}$ (a sum of cubes)
(4) $\boldsymbol{a}^{\mathbf{3}}-\boldsymbol{b}^{\mathbf{3}}$ (a difference of cubes)

Binomials of the form (3) and (4) may not be covered until intermediate algebra.
(1) A sum of squares $\boldsymbol{a}^{\mathbf{2}}+\boldsymbol{b}^{\mathbf{2}}$ is (generally) not factorable
$\left(2 a^{2}+2 b^{2}\right.$ and $a^{6}+b^{6}$ are factorable.)
(2) A difference of squares

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

is extremely important. If your career involves a scientific/mathematical endeavor, you will use factoring a difference of squares when you least expect it.
(3) A sum of cubes (in general odd powers) becomes

$$
a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right)
$$

If the first set of parentheses has a + sign, then the second has alternating signs.
The following patterns are presented to help you remember factoring a sum of cubes.

$$
\begin{aligned}
& a^{5}+b^{5}=(a+b)\left(a^{4}{ }_{-}-a^{3}{ }_{-}+a^{2}{ }_{-}-a^{1}{ }_{-}+a^{0}\right) \text { recall } a^{0}=1 \\
& =(a+b)\left(a^{4} b^{0}-a^{3} b^{1}+a^{2} b^{2}-a b^{3}+b^{4}\right) \quad \text { See the pattern } \\
& \text { of exponents? } \\
& =(a+b)\left(a^{4}-a^{3} b+a^{2} b^{2}-a b^{3}+b^{4}\right) \\
& a^{7}+b^{7} \\
& =(a+b)\left(a^{6}-a^{5}{ }_{-}+a^{4}{ }_{-}-a^{3}{ }_{-}+a^{2}{ }_{-}-a^{1}+a^{0}\right) \\
& =(a+b)\left(a^{6} b^{0}-a^{5} b^{1}+a^{4} b^{2}-a^{3} b^{3}+a^{2} b^{4}-a b^{5}+b^{6}\right) \\
& =(a+b)\left(a^{6} \quad-a^{5} b+a^{4} b^{2}-a^{3} b^{3}+a^{2} b^{4}-a b^{5}+b^{6}\right)
\end{aligned}
$$

(4) A difference of cubes (in general odd powers) becomes

$$
a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
$$

If the first set of parentheses has a - sign, then the second has all + signs.
The following patterns are presented to help you remember factoring a sum of cubes.

$$
\begin{aligned}
& a^{5}-b^{5} \\
= & (a-b)\left(a^{4}+a^{3}+a^{2}+a^{1}+a^{0}\right) \quad \text { remember } a^{0}=1 \\
= & (a-b)\left(a^{4} b^{0}+a^{3} b^{1}+a^{2} b^{2}+a b^{3}+b^{4}\right) \quad \text { See pattern of exponents? } \\
= & (a-b)\left(a^{4}+a^{3} b+a^{2} b^{2}+a b^{3}+b^{4}\right) \\
& a^{7}-b^{7} \\
= & (a-b)\left(a^{6}+a^{5}+a^{4}+a^{3}+a^{2}+a^{1}+a^{0}\right) \\
= & (a-b)\left(a^{6} b^{0}+a^{5} b^{1}+a^{4} b^{2}+a^{3} b^{3}+a^{2} b^{4}+a b^{5}+b^{6}\right) \\
= & (a-b)\left(a^{6}+a^{5} b+a^{4} b^{2}+a^{3} b^{3}+a^{2} b^{4}+a b^{5}+b^{6}\right)
\end{aligned}
$$

### 38.3 Examples

## Example 1:

Factor completely $5 x^{4}-405$

## Solution:

$$
\begin{aligned}
5 x^{4}-405 & =\mathbf{5}\left(x^{4}-81\right) & & \text { Always look for GCF } \\
& =5\left[\left(x^{2}\right)^{2}-(9)^{2}\right] & & \text { Difference of two squares } \\
& =5\left(x^{2}+\mathbf{9}\right)\left[x^{2}-\mathbf{9}\right] & & \text { Sum of squares can't be f } \\
& =5\left(x^{2}+\mathbf{9}\right)\left[x^{2}-3^{2}\right] & & \text { Difference of two squares } \\
& =5\left(x^{2}+\mathbf{9}\right)(x+3)(x-3) & &
\end{aligned}
$$

Example 2:
Factor completely $5 x(x+5)^{2}-80 x$

## Solution:

$$
\begin{array}{rll} 
& 5 x(x+5)^{2}-80 x & \\
= & 5 x\left[(x+5)^{2}-16\right] & \text { Always look for GCF } \\
= & 5 x\left[(x+5)^{2}-4^{2}\right] & \text { Difference of two squares } \\
= & 5 x[(x+5)+4][(x+5)-4] & \\
= & 5 x(x+9)(x+1) &
\end{array}
$$

Example 3:
You own a square plot of land. You reserve a square area of the plot in the south-east corner for construction of a residence. The remaining area, used for farmland (plot minus residence), is to be $4,000 \mathrm{ft}^{2}$. The distance from the north-east corner of the plot to the north-east corner of the residence is to be 40 ft .
What are the dimensions of the plot?


## Solution:

Let $\boldsymbol{x}$ be the side of the square plot and $\boldsymbol{y}$ the side of the square residence. Then $\boldsymbol{x}^{\mathbf{2}}$ is the area of the plot and $\boldsymbol{y}^{\mathbf{2}}$ is the area of the residence. Thus

$$
x^{2}-y^{2}=4,000
$$

The distance from the north-east corner of the plot to the north-east corner of the residence is to be $\mathbf{4 0} \mathrm{ft}$ which means

$$
x-y=40
$$

$$
\begin{array}{rlrl}
x^{2}-y^{2} & =4,000 & \\
(x-y)(x+y) & =4,000 & & \text { Difference of squares } \\
(40)(x+y) & =4,000 & & \text { since } x-y=40 \\
x+y & =\mathbf{1 0 0} & \\
x-y & =\mathbf{4 0} & & \\
2 \boldsymbol{x} & =\mathbf{1 4 0} & & \text { Add the previous two equations } \\
x & =\mathbf{7 0} & & \text { Divide by } 2
\end{array}
$$

The plot of land is $\mathbf{7 0}$ by $\mathbf{7 0} \mathrm{ft}$.
Example 4:
Factor completely $\boldsymbol{x}^{\mathbf{6}}-\mathbf{1}$

## Solution:

Method 1:

$$
\begin{array}{rll} 
& x^{6}-1 & \\
= & x^{6}-1^{6} & \\
= & \left(x^{3}\right)^{2}-\left(1^{3}\right)^{2} & \text { difference of squares } \\
=\left(x^{3}+1^{3}\right)\left(x^{3}-1^{3}\right) & \text { sum/difference of cubes } \\
=(x+1)\left[x^{2}-x(1)+1^{2}\right](x-1)\left[x^{2}+x(1)+1^{2}\right] & a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right) \\
=(x+1)\left(x^{2}-x+1\right)(x-1)\left(x^{2}+x+1\right) & a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right.
\end{array}
$$

Method 2:

$$
\begin{array}{rlr} 
& x^{6}-1 & \\
= & x^{6}-1^{6} & \\
= & \left(x^{2}\right)^{3}-\left(1^{2}\right)^{3} & \\
= & {\left[\left(x^{2}\right)-\left(1^{2}\right)\right]\left[\left(x^{2}\right)^{2}+\left(x^{2}\right)\left(1^{2}\right)+\left(1^{2}\right)^{2}\right]} & \\
=(x+1)(x-1)\left[\left(x^{2}\right)^{2}+x^{2}+1\right] & & \text { difference of cubes } a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right) \\
= & (x+1)(x-1)\left[\left(x^{2}\right)^{2}+x^{2}+1+0\right] & \\
= & (x+1)(x-1)\left[\left(x^{2}\right)^{2}+x^{2}+1+x^{2}-x^{2}\right] & 0=x^{2}-x^{2} \\
= & (x+1)(x-1)\left[\left(x^{2}\right)^{2}+2 x^{2}+1-x^{2}\right] & \text { we of squares } 0 \\
& (x+1)(x-1)\left[\left(x^{2}\right)^{2}+2 x^{2}+1-x^{2}\right] & \text { previous identity in reverse with } a=x^{2}, b=1 \\
= & (x+1)(x-1)\left[\left(x^{2}+1\right)^{2}-\left(x^{2}\right)\right] & \\
= & (x+1)(x-1)\left[\left(x^{2}+1\right)+x\right]\left[\left(x^{2}+1\right)-x\right] \text { difference of two squares } \\
= & (x+1)(x-1)\left(x^{2}+x+1\right)\left(x^{2}-x+1\right) & \\
& \text { same as by method } 1 .
\end{array}
$$

Note: The "trick" of adding 0 was to produce the same result by both methods. If you can come up with this trick by yourselves, you are fantastic.

### 38.4 Exercise 38

1. Factor completely $\mathbf{6} \boldsymbol{x}^{8}-\mathbf{1 , 5 3 6}$
2. Factor completely $(x+5)^{3}-9(x+5)$
3. Factor completely by first using a difference if squares $x^{6}-y^{6}$

## STOP!

1. Factor completely $\mathbf{6} \boldsymbol{x}^{8}-\mathbf{1}, \mathbf{5 3 6}$

## Solution:

$$
\begin{array}{rll} 
& 6 x^{8}-1,536 & \\
= & \mathbf{6}\left(x^{8}-\mathbf{2 5 6}\right) & \text { Always look for GCF } \\
= & 6\left(x^{8}-2^{8}\right) & \\
= & 6\left[\left(x^{4}\right)^{2}-\left(2^{4}\right)^{2}\right] & \text { Difference of two squares } \\
= & 6\left(x^{4}+2^{4}\right)\left(x^{4}-2^{4}\right) & \\
\left.=6\left(x^{4}+2^{4}\right)\left[\left(x^{2}\right)^{2}-\left(2^{2}\right)^{2}\right)\right] & \text { Difference of two squares } \\
= & 6\left(x^{4}+2^{4}\right)\left(x^{2}+2^{2}\right)\left(x^{2}-2^{2}\right) & \text { Difference of two squares } \\
= & 6\left(x^{4}+2^{4}\right)\left(x^{2}+2^{2}\right)(x+2)(x-2) &
\end{array}
$$

2. Factor completely $(x+5)^{3}-\mathbf{9}(x+5)$

## Solution:

$$
\begin{array}{rll} 
& (x+5)^{3}-9(x+5) & \\
= & (x+5)\left[(x+5)^{2}-9\right] & \text { Always look for GCF } \\
= & (x+5)[(x+5)+3][(x+5)-3] & \text { Difference of two squares } \\
= & (x+5)(x+8)(x+2) &
\end{array}
$$

3. Factor completely by first using a difference if squares
$x^{6}-y^{6}$

## Solution:

$$
\begin{aligned}
& x^{6}-y^{6} \\
= & \left(x^{3}\right)^{2}-\left(y^{3}\right)^{2} \\
= & \left(x^{3}+y^{3}\right)\left(x^{3}-y^{3}\right) \\
= & (x+y)\left[x^{2}-x y+y^{2}\right](x-y)\left[x^{2}+x y+y^{2}\right] \\
& \\
& \\
& a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right) \\
& \\
& \\
& \text { and } a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
\end{aligned}
$$

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## Chapter 39

## Factoring trinomials

### 39.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqTs1Pe0L-CLvsgnKXu8nhxu\&feature=view_all

### 39.2 Basics

Recall that a polynomial is a sum or difference of terms (could be in more than one variable) in the form $\boldsymbol{a} \boldsymbol{x}^{\boldsymbol{n}}$ where $\boldsymbol{a}$ is a coefficient and $\boldsymbol{n}$ is a whole number. If a polynomial consists of exactly three terms, then it is a trinomial. The prefix "tri" refers to three, as in tricycle, trilingual, trifocal, tripartite, trimester, ...

In the reduction, addition, subtraction of fractions, it may be necessary to convert trinomials into products by factoring (if possible).

After factoring out the GCF (if any besides 1) we try to factor trinomials of the form $\boldsymbol{a} \boldsymbol{x}^{\mathbf{2}}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}$ where $a>1$. (If $a=1$, the procedure is simpler.)

We are looking for two numbers $\boldsymbol{m}$ and $\boldsymbol{n}$ such that

$$
\begin{aligned}
a x^{2}+b x+c & =\frac{1}{a}(a x-m)(a x-n) \\
& =\frac{1}{a}\left(a^{2} x^{2}-a m x-a n x+m n\right) \\
& =\frac{1}{a}\left(a^{2} x^{2}-a(m+n) x+m n\right) \\
& =a x^{2}-(m+n) x+\frac{1}{a} m n
\end{aligned}
$$

Thus $\boldsymbol{m}+\boldsymbol{n}=-\boldsymbol{b}$ and $\frac{\boldsymbol{m} \boldsymbol{n}}{\boldsymbol{a}}=\boldsymbol{c}$ or $\boldsymbol{m n}=\boldsymbol{a} \boldsymbol{c}$
The above proof can be deduced from looking at www.mathforum.org/library/drmath/view/56442.html

The technique for factoring $\boldsymbol{a} \boldsymbol{x}^{\mathbf{2}}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}$ is that of finding two integers (if they exist) whose product $\boldsymbol{P}=\boldsymbol{a} \boldsymbol{c}$ and whose sum is $\boldsymbol{S}=\boldsymbol{b}$.

Before embarking on the algorithm, decide the sign convention. Are $\boldsymbol{m}$ and $\boldsymbol{n}$ both positive, both negative, or opposites?

Let us write the small number on the left and the large number on the right. The sign of the large number is the sign of $\boldsymbol{b}$.
(case 1)

(case 2)
(case 3)

(case 4)

$$
P=-
$$



For case 1, the small number (on the left) must be positive since a positive (small) number times a positive (large) number is a positive product $\boldsymbol{P}$.
(case $\mathbf{1}$ )

$$
\begin{gathered}
\boldsymbol{P}=+ \\
s=+ \\
s \quad--\sigma^{-} \quad l \\
\\
\\
\\
\end{gathered}
$$

For case 2, the small number (on the left) must be negative since a negative (small) number times a negative (large) number is a positive product $\boldsymbol{P}$.
(case 2)


For case $\mathbf{3}$, the small number (on the left) must be positive since a positive (small) number times a negative (large) number is a negative product $\boldsymbol{P}$.
(case 3)


For case 4, the small number (on the left) must be negative since a negative (small) number times a positive (large) number is a negative product $\boldsymbol{P}$.
(case 4)


Now that we know the signs of the two numbers we seek, systematically work with the divisors (factors) of the product starting with $\mathbf{1}$. The small numbers will be $\mathbf{1}, \mathbf{2}$ (if applicable), $\cdots$.

Some students are good at factoring trinomials quickly by inspection. If you are one of these students, you may think you can skip the development presented here. If you do, how can you convince someone that a particular trinomial cannot be factored using integers?

If the coefficient $\boldsymbol{a}=\mathbf{1}$ in $\boldsymbol{a} \boldsymbol{x}^{\mathbf{2}}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}$, the procedure outlined here can still be applied. If the numbers are small, factorization may be obtained by inspection. See example 7 below.

### 39.3 Examples

Example 1:
Factor $\mathbf{8} x^{2}+\mathbf{2 2 x}+\mathbf{1 5}$

## Solution:

$8=2 \cdot 2 \cdot 2$
What is the GCF? Only 2 (and multiples of $\mathbf{2}$ ) is a factor of $\mathbf{8} .2$ does not divide $\mathbf{1 5}$, so the numerical part of the GCF is $\mathbf{1} . \boldsymbol{x}$ is not a factor of $\mathbf{1 5}$, so the variable part of the GCF is also $\mathbf{1}$.

| $P$ | $=120$ |  |
| ---: | :--- | :--- | :--- |
| $S$ | $=22$ |  |
| $($ case 4$)$ | - | $l$ |
| 1 | 120 |  |
| 2 | 60 |  |
| 3 | 40 |  |
| 4 | 30 |  |
| 5 | 24 |  |
| 6 | 20 |  |
| 8 | 15 |  |
| 10 | 12 | The sum is 22 |

The magic numbers 10 and 12 tell us how to split up the middle (linear) term $\mathbf{2 2 x}$.

$$
\begin{aligned}
8 x^{2}+22 x+15 & =8 x^{2}+10 x+12 x+15 \\
& =2 x(4 x+5)+3(4 x+5) \text { Factor GCF from each pair. } \\
& =(4 x+5)(2 x+3) \quad \text { Factor GCF. }
\end{aligned}
$$

Thus $8 x^{2}+22 x+15=(4 x+5)(2 x+3)$
Example 2:
Factor $8 x^{2}+2 x-15$

## Solution:

$8=2 \cdot 2 \cdot 2$
What is the GCF? Only 2 (and multiples of $\mathbf{2}$ ) is a factor of $\mathbf{8} .2$ does not divide $\mathbf{1 5}$, so the numerical part of the GCF is $\mathbf{1} . \boldsymbol{x}$ is not a factor of $\mathbf{1 5}$, so the variable part of the GCF is also $\mathbf{1}$.

| $\begin{aligned} P & =-120 \\ S & =2\end{aligned}$ |  |  |
| :---: | :---: | :---: |
| $s$ | - | $l$ |
| -1 | 120 |  |
| -2 | 60 |  |
| -3 | 40 |  |
| -4 | 30 |  |
| -5 | 24 |  |
| -6 | 20 |  |
| -8 | 15 |  |
| -10 | 12 | The sum is $\mathbf{2}$ |
|  |  |  |

The magic numbers $\mathbf{- 1 0}$ and $\mathbf{1 2}$ tell us how to split up the middle (linear) term $\mathbf{2 x}$.

$$
\begin{array}{rlr}
8 x^{2}+22 x+15 & =8 x^{2}-10 x+12 x-15 \\
& =2 x(4 x-5)+3(4 x-5) & \text { Factor GCF from each pair. } \\
& =(4 x-5)(2 x+3) & \text { Factor GCF. }
\end{array}
$$

Thus $8 x^{2}+2 x-15=(4 x-5)(2 x+3)$
Example 3:
Factor $8 x^{2}-2 x-15$

## Solution:

$\mathbf{8}=\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2}$ is a factor of $\mathbf{8} .2$ does not divide $\mathbf{1 5}$, so the numerical part of the GCF is $\mathbf{1} . \boldsymbol{x}$ is not a factor of $\mathbf{1 5}$, so the variable part of the GCF is also $\mathbf{1}$.

$$
\begin{aligned}
& P=-120 \\
& S=-2
\end{aligned}
$$

$s$

| $-\rceil-$ |  | $l$ |
| :---: | :---: | :---: |
| $\mathbf{1}$ | $-\mathbf{1 2 0}$ |  |
| $\mathbf{2}$ | $-\mathbf{6 0}$ |  |
| $\mathbf{3}$ | -40 |  |
| $\mathbf{4}$ | $-\mathbf{3 0}$ |  |
| $\mathbf{5}$ | $-\mathbf{2 4}$ |  |
| $\mathbf{6}$ | $-\mathbf{2 0}$ |  |
| $\mathbf{8}$ | $-\mathbf{1 5}$ |  |
| $\mathbf{1 0}$ | $-\mathbf{1 2}$ | The sum is $-\mathbf{2}$ |

The magic numbers $\mathbf{1 0}$ and $\mathbf{- 1 2}$ tell us how to split up the middle (linear) term $\mathbf{- 2 \boldsymbol { x }}$.

$$
\begin{array}{rlr}
8 x^{2}-2 x-15 & =8 x^{2}+10 x-12 x-15 \\
& =2 x(4 x+5)-3(4 x+5) & \text { Factor GCF from each pair. } \\
& =(4 x+5)(2 x-3) & \text { Factor GCF. }
\end{array}
$$

Thus $8 x^{2}-2 x-15=(4 x+5)(2 x-3)$
Example 4:
Factor $8 x^{2}-22 x+15$

## Solution:

$8=2 \cdot 2 \cdot 2$
What is the GCF? Only 2 (and multiples of $\mathbf{2}$ ) is a factor of $\mathbf{8} .2$ does not divide $\mathbf{1 5}$, so the numerical part of the GCF is $\mathbf{1}$. $\boldsymbol{x}$ is not a factor of $\mathbf{1 5}$, so the variable part of the GCF is also $\mathbf{1}$.

$$
\begin{aligned}
& P=120 \\
& S=-22 \\
& \\
& -2 \mid-60 \\
& -3 \mid-40 \\
& \begin{array}{l|l}
-4 & -30
\end{array} \\
& \begin{array}{l|l}
-5 & -24
\end{array} \\
& -6 \mid-20 \\
& \begin{array}{l|l}
-8 & -15
\end{array} \\
& -\mathbf{1 0} \mid-\mathbf{1 2} \text { The sum is } \mathbf{- 2 2}
\end{aligned}
$$

The magic numbers $\mathbf{- 1 0}$ and $\mathbf{- 1 2}$ tell us how to split up the middle (linear) term $\mathbf{- 2 2 x}$.

$$
\begin{array}{rll} 
& 8 x^{2}-22 x+15 & \\
= & 8 x^{2}-10 x-12 x+15 & \\
= & 2 x(4 x-5)-3(4 x-5) & \\
=(4 x-5)(2 x-3) & \text { Factor GCF from each pair. } \\
= & \text { Factor GCF. }
\end{array}
$$

Thus $8 x^{2}-2 x-15=(4 x-5)(2 x-3)$
Example 5:
Factor completely (if possible) $\mathbf{5 4} \boldsymbol{x}^{\mathbf{3}}-\mathbf{8 1} \boldsymbol{x}^{\mathbf{2}}-\mathbf{1 0 5 x}$

## Solution:

$54=2 \cdot 3 \cdot 3 \cdot 3$
What is the GCF? 2 is a factor of $\mathbf{5 4}$ but not of $\mathbf{8 1 .} \mathbf{3}$ is a factor of $\mathbf{5 4}, \mathbf{8 1}$, and $\mathbf{1 0 5}$. $\mathbf{9}$ is a factor of $\mathbf{5 4}$, and $\mathbf{8 1}$, but not 105. The numerical part of the GCF is $\mathbf{3} . \boldsymbol{x}$ is a factor of $\boldsymbol{x}^{\mathbf{3}}, \boldsymbol{x}^{\mathbf{2}}$, and $\boldsymbol{x}$ so the variable part of the GCF. The GCF is $3 \boldsymbol{x}$.
$\left(18 \boldsymbol{x}^{\mathbf{2}}-\mathbf{2 7} \boldsymbol{x}-\mathbf{3 5}\right.$ ) (temporaryly omit the GCF)

```
P=(18)(-35) = -630
    P=-630
        S=-27
s
    M
```

The magic numbers $\mathbf{1 5}$ and $\mathbf{- 4 2}$ tell us how to split up the middle (linear) term $\mathbf{- 2 7 x}$.

$$
\begin{array}{rlr}
18 x^{2}-27 x-35 & =18 x^{2}+15 x-42 x-35 \\
& =3 x(6 x+5)-7(6 x+5) & \\
& =(6 x+5)(3 x-7) & \text { Factor GCF from each pair. } \\
& =(6 a c t o r \text { GCF. }
\end{array}
$$

Thus $54 x^{3}-81 x^{2}-105 x=3 x(6 x+5)(3 x-7)$
Example 6:
Factor completely (if possible) $\mathbf{1 2} \boldsymbol{x}^{\mathbf{2}}+\mathbf{2 x}-\mathbf{3 5}$

## Solution:

$12=2 \cdot 3 \cdot 3$
$\mathbf{2}$ is not a divisor of $\mathbf{3 5}$. Thus $\mathbf{1}$ is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the third term. The variable part of the GCF is $\mathbf{1}$.

```
P=(12)(-35) = -420
    P=-420
        S=2
s ---\Gamma-l
    -1 420
    -2 
    -3 | 140
    -4 105
    -5 | 84
    -6 % 70
    -7 60 skip -8, -9
    -10 | 42 skip -11
    -12 | 35
    -14 | 30 skip -13
    -15 28 skip -16, -17, -18, -19. The sum is 12.
    -20 
```

The magic numbers do not exist. This trinomial cannot be factored using integers. We covered all possibilities.

How long would you have spent looking for non-existing factors?

Some trinomials are perfect squares. Recall that
$(a \pm b)^{2}=a^{2} \pm 2 a b+b^{2}$ and apply this identity in reverse.
Example 7:
Factor completely (if possible) $\boldsymbol{x}^{\mathbf{2}}+\mathbf{1 2 x}+\mathbf{3 5}$

## Solution:

The GCF is $\mathbf{1}$.
$\mathbf{3 5}=\mathbf{5} \cdot \mathbf{7}$ and $5+\mathbf{7}=12.5$ and $\mathbf{7}$ are the magic numbers.
$x^{2}+12 x+35=x^{2}+5 x+7 x+35$
$=x(x+5)+7(x+5)$ Factor GCF from each pair.
$=(x+5)(x+7) \quad$ Factor GCF.
Thus $x^{2}+12 x+35=(x+5)(x+7)$
Example 8:
Factor completely (if possible) $\boldsymbol{x}^{\mathbf{2}}-\mathbf{6 x}-\mathbf{1 6}$

## Solution:

## The GCF is $\mathbf{1}$.

The constant is negative, so the two magic numbers are opposites. The coefficient of the linear term is negative, so the larger number (in absolute value) is negative. $\mathbf{- 8}$ and $\mathbf{2}$ seem likely candidates for the magic numbers.
$x^{2}-6 x-16=(x-8)(x+2)$
Example 9:
Factor completely (if possible) $\boldsymbol{x}^{2}+\mathbf{3 x}-\mathbf{5 4}$

## Solution:

The GCF is $\mathbf{1}$.
The constant is negative, so the two magic numbers are opposites. The coefficient of the linear term is positive, so the larger number (in absolute value) is positive. $\mathbf{9}$ and $\mathbf{- 6}$ seem likely candidates for the magic numbers.

$$
x^{2}+3 x-54=(x+9)(x-6)
$$

Example 10:
Factor completely (if possible) $\boldsymbol{x}^{\mathbf{2}}-\mathbf{2 0} \boldsymbol{x}+\mathbf{6 4}$

## Solution:

## The GCF is $\mathbf{1}$.

The constant is positive, so the two magic numbers are of the same sign. The coefficient of the linear term is negative, so the numbers are both negative. $\mathbf{- 1 6}$ and $\mathbf{- 4}$ seem likely candidates for the magic numbers.
$x^{2}-20 x+64=(x-16)(x-4)$
Example 11:
Factor completely (if possible) $\mathbf{9} \boldsymbol{x}^{4}+\mathbf{1 8} \boldsymbol{x}^{\mathbf{2}}-\mathbf{1 3 5}$

## Solution:

$9=3 \cdot 3$
What is the GCF? 3 is a factor of $\mathbf{9}, \mathbf{1 8}$, and $\mathbf{1 3 5 . 9}$ is a factor of $\mathbf{5 4}, \mathbf{1 8}$, and $\mathbf{1 3 5}$. The numerical part of the GCF is $9 . \boldsymbol{x}$ is a factor of $\boldsymbol{x}^{4}, \boldsymbol{x}^{2}$, but not 135 so the variable part of the GCF is $\mathbf{1}$. The GCF is $\mathbf{9}$.
$9 x^{4}+18 x^{2}-135=9\left(x^{4}+2 x^{2}-15\right)$
The constant is negative, so the two magic numbers are opposites. The coefficient of the linear term is positive, so the larger number (in absolute value) is positive. $\mathbf{- 3}$ and $\mathbf{5}$ seem likely candidates for the magic numbers.
$9\left(x^{4}+2 x^{2}-15\right)=9\left(x^{2}-3\right)\left(x^{2}+5\right)$

### 39.4 Exercise 39

1. Factor completely (if possible) $\mathbf{1 2} \boldsymbol{x}^{\mathbf{2}}-\mathbf{2 2 x}-\mathbf{1 4}$
2. Factor completely (if possible) $12 x^{2}-\mathbf{2 2 x}-144$
3. Factor completely (if possible) $\mathbf{2 4} \boldsymbol{x}^{\mathbf{2}}-\mathbf{9 8 x}+\mathbf{6 5}$
4. Factor completely (if possible) $\mathbf{4 8} \boldsymbol{x}^{2}-\mathbf{2 1 6 x}+\mathbf{2 4 3}$.
5. Factor completely (if possible) $\mathbf{6 3} \boldsymbol{x}^{\mathbf{2}}+\mathbf{1 4 7 x}+\mathbf{3 4 3}$.
6. Factor completely (if possible) $\mathbf{2} \boldsymbol{x}^{\mathbf{2}}-\mathbf{3 6 x}+\mathbf{1 6 0}$
7. Factor completely (if possible) $\boldsymbol{x}^{2}-\mathbf{1 2 x}-108$
8. Factor completely (if possible) $\boldsymbol{x}^{\mathbf{2}}-\mathbf{1 8 x}+\mathbf{2 4 3}$
9. Factor completely (if possible) $\boldsymbol{x}^{8}-\mathbf{3 2} \boldsymbol{x}^{\mathbf{4}}+\mathbf{2 5 6}$.

## STOP!

1. Factor completely (if possible) $\mathbf{1 2} \boldsymbol{x}^{2}-\mathbf{2 2 x}-\mathbf{1 4}$

## Solution:

$12=2 \cdot 2 \cdot 3$
What is the GCF? 2 is a factor of 12. 22, and 14. 4 is a factor of $\mathbf{1 2}$ but not of $\mathbf{2 2} .2$ is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the last term, so the variable part of the GCF is $\mathbf{1}$. The GCF is $\mathbf{2}$.

$$
\begin{aligned}
& 12 x^{2}-22 x-14=2\left(6 x^{2}-11 x-7\right) \\
& P=(6)(-7)=-42
\end{aligned}
$$

\[

\]

The magic numbers $\mathbf{3}$ and $\mathbf{- 1 4}$ tell us how to split up the middle (linear) term $\mathbf{- 1 1 \boldsymbol { x }}$.

$$
\begin{array}{rll} 
& 12 x^{2}-22 x-14 & \\
= & 2\left(6 x^{2}-11 x-7\right) & \\
=2\left(6 x^{2}+3 x-14 x-7\right) & & \\
=2[3 x(2 x+1)-7(2 x+1)] & & \text { Factor GCF from each pair. } \\
=2(2 x+1)(3 x-7) & & \text { Factor GCF. }
\end{array}
$$

Thus $12 x^{2}-22 x-14=2(2 x+1)(3 x-7)$
2. Factor completely (if possible) $\mathbf{1 2} \boldsymbol{x}^{\mathbf{2}}-\mathbf{2 2 x}-\mathbf{1 4 4}$

## Solution:

$12=2 \cdot 2 \cdot 3$
What is the GCF? 2 is a factor of 12.22 , and 144.4 is a factor of 12 but not of 22.2 is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the last term, so the variable part of the GCF is 1 . The GCF is 2 .

$$
\begin{aligned}
& 12 x^{2}-22 x-144=2\left(6 x^{2}-11 x-72\right) \\
& P=(6)(-72)=-432
\end{aligned}
$$

$$
\begin{gathered}
P=-432 \\
S=-11
\end{gathered}
$$

$s$

|  |  | $l$ |
| :---: | :---: | :---: |
| 1 | \| -432 |  |
| 2 | \|-216 |  |
| 3 | \| - 144 | sum $=-11$ |
| 4 | \| - 108 |  |
| 6 | \| -72 |  |
| 8 | \| - 56 |  |
| 9 | \| -48 |  |
| 12 | \| -36 |  |
| 16 | \| -27 | The sum is $\mathbf{- 1 1}$ |

The magic numbers $\mathbf{1 6}$ and $\mathbf{- 2 7}$ tell us how to split up the middle (linear) term $\mathbf{- 1 1 x}$.

$$
\begin{array}{rll} 
& 12 x^{2}-22 x-144 & \\
= & 2\left(6 x^{2}-11 x-72\right) & \\
= & 2\left(6 x^{2}+16 x-27 x-72\right) & \\
= & 2[2 x(3 x+8)-9(3 x+8)] & \\
= & \text { Factor GCF from each pair. } \\
=2(3 x+8)(2 x-9) & \text { Factor GCF. }
\end{array}
$$

Thus $12 x^{2}-22 x-144=2(3 x+8)(2 x-9)$
3. Factor completely (if possible) $\mathbf{2 4} x^{2}-98 x+65$

## Solution:

$24=2 \cdot 2 \cdot 2 \cdot 3$
What is the GCF? 2 is a factor of 24.98 , but not of 65.3 is a factor of 24 but not of 98.1 is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the last term, so the variable part of the GCF is $\mathbf{1}$. The GCF is $\mathbf{1}$.

$$
\begin{aligned}
& P=(24)(65)=1,560 \\
& P=1560 \\
& S=-98 \\
& s
\end{aligned}
$$

$$
\begin{aligned}
& -2 \mid-780 \\
& -3 \mid-520 \\
& -4 \mid-390 \\
& \begin{array}{l|l}
-5 & -312
\end{array} \\
& -6 \mid-260 \\
& -10 \mid-156 \\
& -12 \text { | }-130 \\
& -15 \mid-104 \\
& -\mathbf{2 0} \mid-78 \text { The sum is }-\mathbf{9 8}
\end{aligned}
$$

The magic numbers $\mathbf{- 2 0}$ and $\mathbf{- 7 8}$ tell us how to split up the middle (linear) term $\mathbf{- 9 8 x}$.

$$
\begin{array}{rll} 
& 24 x^{2}-98 x+65 & \\
= & 24 x^{2}-20 x-78 x+65 & \\
= & 4 x(6 x-5)-13(6 x-5) & \text { Factor GCF from each pair. } \\
= & (6 x-5)(4 x-13) & \text { Factor GCF. }
\end{array}
$$

Thus $\left.24 x^{2}-98 x+65=(6 x-5)(4 x-13)\right)$
4. Factor completely (if possible) $48 \boldsymbol{x}^{2}-\mathbf{2 1 6 x}+\mathbf{2 4 3}$.

## Solution:

$48=2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
What is the GCF? 2 is a factor of 48. 216, but not of 243.3 is a factor of 24,216 , and 243.3 is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the last term, so the variable part of the GCF is 1 . The GCF is $\mathbf{3}$.

$$
\begin{aligned}
& 48 x^{2}-216 x+243=3\left(16 x^{2}-72 x+81\right) \\
& 48 x^{2}-216 x+243 \\
&= 3\left(16 x^{2}-72 x+81\right. \\
&= 3\left[(4 x)^{2}-2(4 x)(9)+(9)^{2}\right] \quad(a-b)^{2}=a^{2}-2 a b+b^{2} \\
&= 3(4 x-9)^{2}
\end{aligned}
$$

5. Factor completely (if possible) $\mathbf{6 3} \boldsymbol{x}^{\mathbf{2}}+\mathbf{1 4 7 x}+\mathbf{3 4 3}$.

## Solution:

$63=3 \cdot 3 \cdot 7$.
What is the GCF? 3 is a factor of 63.147 , but not of $\mathbf{3 4 3 .} 7$ is a factor of 63,147 , and 343.7 is the numerical part of the GCF. $\boldsymbol{x}$ is not a factor of the last term, so the variable part of the GCF is 1. The GCF is 7 .

$$
\begin{aligned}
& 63 x^{2}+147 x+343=7\left(9 x^{2}+21 x+49\right) \\
&=7\left[(3 x)^{2}+(3 x)(7)+(7)^{2}\right] \\
&(a+b)^{2}=a^{2}+2 a b+b^{2}
\end{aligned}
$$

The trinomial is not a perfect square.
The double product term $2(3 x)(7)$ is missing.
Try factoring with product/sum method.
6. Factor completely (if possible) $\mathbf{2} \boldsymbol{x}^{\mathbf{2}}-\mathbf{3 6 x}+\mathbf{1 6 0}$

## Solution:

The GCF is 2.
$2 x^{2}-36 x+160=2\left(x^{2}-18 x+80\right)$
The constant is positive, so the two magic numbers are of the same sign. The coefficient of the linear term is negative, so the numbers are both negative. $\mathbf{- 1 0}$ and $\mathbf{- 8}$ seem likely candidates for the magic numbers.
$2 x^{2}-36 x+160=2\left(x^{2}-18 x+80\right)=2(x-10)(x-8)$
7. Factor completely (if possible) $\boldsymbol{x}^{2}-\mathbf{1 2 x}-108$

## Solution:

The GCF is $\mathbf{1}$.
The constant is negative, so the two magic numbers are opposites. The coefficient of the linear term is negative, so the larger number (in absolute value) is negative. $\mathbf{- 1 8}$ and $\mathbf{6}$ seem likely candidates for the magic numbers.

$$
x^{2}-12 x-108=(x-18)(x+6)
$$

8. Factor completely (if possible) $\boldsymbol{x}^{\mathbf{2}}-\mathbf{1 8 x}+\mathbf{2 4 3}$

## Solution:

The GCF is $\mathbf{1}$.
The constant is positive, so the two magic numbers are of the same sign. The coefficient of the linear term is negative, so the numbers are both negative. $\mathbf{- 2 7}$ and $\mathbf{- 9}$ seem likely candidates for the magic numbers.

$$
x^{2}-18 x+243 ?=?(x-27)(x-9)=x^{2}-36 x+243
$$

What are the factors?

We cycled through all the factors of $\mathbf{2 4 3}$. The given trinomial cannot be factored. No two of the above factors add up to $\mathbf{- 1 8}$.
9. Factor completely (if possible) $\boldsymbol{x}^{\mathbf{8}}-\mathbf{3 2} \boldsymbol{x}^{\mathbf{4}}+\mathbf{2 5 6}$.

## Solution:

The GCF is $\mathbf{1}$.

$$
x^{8}-32 x^{4}+256
$$

$$
=\left(x^{4}\right)^{2}-2(16) x^{4}+(16)^{2}
$$

$$
=\left(x^{4}-16\right)^{2} \quad(a-b)^{2}=a^{2}-2 a b+b^{2}
$$

$$
\left.=\left[\left(x^{2}\right)^{2}-(4)^{2}\right)\right]^{2} \quad \text { Difference of squares }
$$

$=\left[\left(x^{2}+4\right)\left(x^{2}-4\right)\right]^{2} \quad$ Difference of squares
$=\left[\left(x^{2}+4\right)(x+2)(x-2)\right]^{2}$

$$
\begin{aligned}
& P=(243)(1)=243=3^{5} \\
& P=243 \\
& S=-18 \\
& s
\end{aligned}
$$

## Chapter 40

# Factoring Perfect Square Trinomials 

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### 40.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqRKvrdhQv6m2qZ7Bj5F3CV-\&feature=view_all

### 40.2 Basics

We have already seen that $(a \pm b)^{2}=a^{2} \pm \mathbf{a} a b+b^{2}$

### 40.3 Examples

Example 1:
Factor completely (if possible) $\mathbf{2 5} \boldsymbol{x}^{\mathbf{2}}-\mathbf{7 0} \boldsymbol{x} \boldsymbol{y}+\mathbf{4 9} \boldsymbol{y}^{\mathbf{2}}$.

## Solution:

The GCF is 1 .

$$
\begin{aligned}
25 x^{2}-70 x y+49 y^{2} & =(5 x)^{2}-2(5 x)(7 y)+(7 y)^{2} \\
& =(5 x-7 y)^{2}
\end{aligned}
$$

Example 2:
Factor completely (if possible) $\mathbf{7 5} \boldsymbol{x}^{\mathbf{5}}-\mathbf{1 2 0} \boldsymbol{x}^{\mathbf{3}} \boldsymbol{y}+\mathbf{4 8 x} \boldsymbol{y ^ { 2 }}$.

## Solution:

$75=3 \cdot 5 \cdot 5$
What is the GCF? $\mathbf{3}$ is a factor of $\mathbf{7 5}, \mathbf{1 2 0}, \mathbf{4 8} .5$ is a factor of $\mathbf{7 5}, \mathbf{1 2 0}$, but not $\mathbf{4 8}$. Thus $\mathbf{3}$ is the numerical part of the GCF. $\boldsymbol{x}$ is the variable part of the GCF. $\boldsymbol{y}$ does not appear in the first term. The GCF is $\mathbf{5 x}$.

$$
\begin{aligned}
& 75 x^{5}-120 x^{3} y+48 x y^{2}=3 x\left(25 x^{4}-40 x^{2} y+16 y^{2}\right) \\
& \left.3 x\left(25 x^{4}-40 x^{2} y+16 y^{2}\right)=3 x\left[\left(5 x^{2}\right)^{2}-2\left(5 x^{2}\right)(4 y)+(4 y)^{2}\right)\right] \\
& =3 x\left(5 x^{2}-4 y\right)^{2}
\end{aligned}
$$

Example 3:
Factor completely (if possible) $x^{2}+18 x+81-y^{2}$.

## Solution:

The GCF is 1. The technique for factoring four terms is pairing. Unfortunately pairing does not lead to success. Notice that the first three terms make up a perfect square.

$$
\begin{aligned}
x^{2}+18 x+81-y^{2} & =(x)^{2}+2(x)(9)+(9)^{2}-y^{2} \\
& =(x+9)^{2}-y^{2} \quad \text { difference of squares } \\
& =(x+9+y)(x+9-y)
\end{aligned}
$$

### 40.4 Exercise 40

1. Factor completely (if possible) $\mathbf{3 6} \boldsymbol{x}^{2}+\mathbf{1 3 2 x} y+\mathbf{1 2 1} \boldsymbol{y}^{\mathbf{2}}$.
2. Factor completely (if possible) $\mathbf{9 8} \boldsymbol{a}^{\mathbf{7}}-\mathbf{1 1 2}^{\mathbf{5}} \boldsymbol{b}+\mathbf{3 2 a}^{\mathbf{3}} \boldsymbol{b}^{\mathbf{2}}$.
3. Factor completely (if possible) $\boldsymbol{u}^{4}-\boldsymbol{v}^{2}+\mathbf{1 0 v}-\mathbf{2 5}$.

## STOP!

1. Factor completely (if possible) $\mathbf{3 6} \boldsymbol{x}^{\mathbf{2}}+\mathbf{1 3 2 x} \boldsymbol{y}+\mathbf{1 2 1} \boldsymbol{y}^{\mathbf{2}}$.

## Solution:

The GCF is 1 .

$$
\begin{aligned}
36 x^{2}+132 x y+121 y^{2} & =(6 x)^{2}+2(6 x)(11 y)+(11 y)^{2} \\
& =(6 x+11 y)^{2}
\end{aligned}
$$

2. Factor completely (if possible) $\mathbf{9 8} \boldsymbol{a}^{\mathbf{7}}-\mathbf{1 1 2} \boldsymbol{a}^{\mathbf{5}} \boldsymbol{b}+\mathbf{3 2} \boldsymbol{a}^{\mathbf{3}} \boldsymbol{b}^{\mathbf{2}}$.

## Solution:

$98=2 \cdot 7 \cdot 7$

What is the GCF? $\mathbf{2}$ is a factor of $\mathbf{9 8}, \mathbf{1 1 2}$, and $\mathbf{3 2}$. $\mathbf{7}$ is not a factor of $\mathbf{7 5}$. Thus $\mathbf{2}$ is the numerical part of the GCF. $\boldsymbol{a}^{\mathbf{3}}$ is the variable part of the GCF. $\boldsymbol{b}$ does not appear in the first term. The GCF is $2 a^{3}$.

$$
\begin{aligned}
& 98 a^{7}-112 a^{5} b+32 a^{3} b^{2}=2 a^{3}\left(49 a^{4}-56 a^{2} b+16 b^{2}\right) \\
& 2 a^{3}\left(49 a^{4}-56 a^{2} b+16 b^{2}\right)= \\
& =2 a^{3}\left[\left(7 a^{2}\right)^{2}-2\left(7 a^{2}\right)(4 b)+(4 b)^{2}\right. \\
& =
\end{aligned}{2 a^{3}\left(7 a^{2}-4 b\right)^{2}}^{2} .
$$

3. Factor completely (if possible) $\boldsymbol{u}^{\mathbf{4}}-\boldsymbol{v}^{\mathbf{2}}+\mathbf{1 0 v}-\mathbf{2 5}$.

## Solution:

The GCF is $\mathbf{1}$. The technique for factoring four terms is pairing. Unfortunately pairing does not lead to success. Notice that the last three terms make up a perfect square (after factoring $\mathbf{- 1}$ ).

$$
\begin{aligned}
u^{4}-v^{2}+10 v-25 & =u^{4}-\left(v^{2}-10 v+25\right) \\
& =u^{4}-\left[(v)^{2}-2(v)(5)+(5)^{2}\right] \\
& =\left(u^{2}\right)^{2}-(v-5)^{2} \quad \text { difference of squares, } \\
& =[u-(v-5)][u+(v-5)] \\
& =(u-v+5)(u+v-5)
\end{aligned}
$$

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## Chapter 41

## General Strategy for Factoring

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### 41.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqQbwk6TjaEGhyWDqTLkmAzB\&feature=view_all


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## Chapter 42

## Solving Quadratic Equations by Factoring

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### 42.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqR24jhxL_XxpTTrdCJJUnw0\&feature=view_all

### 42.2 Basics

This exposé is for special cookbook equations. The left member of the equation $\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}$ is supposed to be factorable.

The key to the solution lies in setting a product equal to $\mathbf{0}$. Then any factor can be zero.
$\mathbf{0} \cdot \mathbf{3} \cdot \boldsymbol{x}=\mathbf{0}$ regardless of the value of $\boldsymbol{x}$.
Evaluate $(x-4)(2 x+5)$ for $x=4$ and $x=-\frac{5}{2}$.
$(4-4)(2 \cdot 4+5)=0 \cdot 13=0$
$\left(-\frac{5}{2}-4\right)\left[2\left(-\frac{5}{2}\right)+5\right]=\frac{-13}{2} \cdot 0=0$

### 42.3 Examples

Example 1:
Solve $5 x(x-9)(x+0.7)=0$

## Solution:

$5 x(x-9)(x+0.7)=0$ if
$\boldsymbol{x}=\mathbf{0}$, and
$\boldsymbol{x}-\mathbf{9}=\mathbf{0}$ or $\boldsymbol{x}=\mathbf{9}$, and
$\boldsymbol{x}+0.7=0$ or $\boldsymbol{x}=-0.7$,
Example 2
Solve $5 x^{2}-3 x=0$.

## Solution:

$$
5 x^{2}-3 x=x(5 x-3)=0
$$

Either $\boldsymbol{x}=\mathbf{0}$ or $\boldsymbol{x}=\frac{\mathbf{3}}{\mathbf{5}}$
Example 3
Solve $x^{2}-3 x=10$.

## Solution:

$$
\begin{aligned}
x^{2}-3 x & =10 \\
x^{2}-3 x-10 & =0 \\
(x-5)(x+2) & =0
\end{aligned}
$$

Either $\boldsymbol{x}=\mathbf{5}$ or $\boldsymbol{x}=\mathbf{- 2}$.
Example 4
Solve $x(x+8)=-15$.

## Solution:

$$
\begin{aligned}
x(x+8) & =-15 \\
x^{2}+8 x+15 & =0 \\
(x+3)(x+5) & =0
\end{aligned}
$$

Either $\boldsymbol{x}=\mathbf{- 5}$ or $\boldsymbol{x}=\mathbf{- 3}$.

Example 5:
A sheet of paper measures 11 by 8 inches. A uniform border surrounds the printed material. What is the size of the uniform border if the printed material is $54 \mathrm{in}^{2}$.

## Solution:

Let $\boldsymbol{x}$ be the measurement of the uniform border. Then
$(11-2 x)(8-2 x)=4 x^{2}-38 x+88$ is the area of the printed material.

$$
\begin{aligned}
4 x^{2}-38 x+88 & =54 \\
4 x^{2}-38 x+34 & =0 \\
2\left(2 x^{2}-19 x+17\right) & =0 \\
2\left(2 x^{2}-2 x-17 x+17\right) & =0 \\
2[2 x(x-1)-17(x-1) & =0 \\
2(x-1)(2 x-17) & =0
\end{aligned}
$$

Thus $\boldsymbol{x}=1$ or $\boldsymbol{x}=8.5$. This latter result cannot be a solution because it is larger than the width of the page.

The border is $\mathbf{1}$ inch.

### 42.4 Exercise 42

1. Solve $8 x^{2}(3 x-7)(x-0.2)=0$
2. Solve $4(x+6)^{2}-11(x+6)=0$.
3. Solve $30 x^{2}-38 x=20$.
4. Solve $\boldsymbol{x}(\boldsymbol{x}-7)=\mathbf{3 0}$.
5. Your rectangular flower bed measures 20 by 5 ft . A uniform walkway around the flower bed increases its area by $\mathbf{3 5 0} \mathrm{ft}^{2}$. Find the dimensions of the new increased rectangle.

## $S T \cap P!$

1. Solve $8 x^{2}(3 x-7)(x-0.2)=0$

Solution:
$8 x^{2}(3 x-7)(x-0.2)=0$ if
$\boldsymbol{x}=\mathbf{0}$, and
$3 x-7=0$ or $x=\frac{7}{3}$, and
$\boldsymbol{x}-0.2=0$ or $\boldsymbol{x}=\mathbf{0 . 2}$,
2. Solve $4(x+6)^{2}-11(x+6)=0$.

## Solution:

$$
\begin{aligned}
& 4(x+6)^{2}-11(x+6)=0 \\
& (x+6)[4(x+6)-11]=0
\end{aligned}
$$

Either $\boldsymbol{x}=\mathbf{- 6}$ or

$$
\begin{aligned}
4(x+6)-11 & =0 \\
4 x+24-11 & =0 \\
4 x+13 & =0 \\
x & =\frac{-13}{4}
\end{aligned}
$$

3. Solve $30 x^{2}-38 x=20$.

Solution:

$$
\begin{aligned}
30 x^{2}-38 x & =20 \\
2\left(15 x^{2}-19 x\right) & =2(10) \\
15 x^{2}-19 x-10 & =0 \\
15 x^{2}+6 x-25 x-10 & =0 \\
(3 x-5)(5 x+2) & =0
\end{aligned}
$$

Either $\boldsymbol{x}=\frac{\mathbf{5}}{\mathbf{3}}$ or $\boldsymbol{x}=-\frac{2}{\mathbf{5}}$.
4. Solve $\boldsymbol{x}(\boldsymbol{x}-7)=\mathbf{3 0}$.

## Solution:

$$
\begin{aligned}
x(x-7) & =30 \\
x^{2}-7 x & =30 \\
x^{2}-7 x-30 & =0 \\
(x-10)(x+3) & =0
\end{aligned}
$$

Either $\boldsymbol{x}=\mathbf{1 0}$ or $\boldsymbol{x}=\mathbf{- 3}$.
5. Your rectangular flower bed measures 20 by 5 ft . A uniform walkway around the flower bed increases its area by $\mathbf{3 5 0} \mathrm{ft}^{2}$. Find the dimensions of the new increased rectangle.

## Solution:

Let $\boldsymbol{x}$ be the measurement of twice the uniform walkway. Then
$(20+x)(5+x)=x^{2}+25 x+100$ is the area of the new rectangle.

$$
\begin{aligned}
x^{2}+25 x+100-100 & =350 \\
x^{2}+25 x & =350 \\
x^{2}+25 x-350 & =0 \\
(x-10)(x+35) & =0
\end{aligned}
$$

Thus $\boldsymbol{x}=10$ or $\boldsymbol{x}=\mathbf{3 5}$. This latter result cannot be a solution because $(\mathbf{3 5}+\mathbf{2 0})(35+5)=\mathbf{2 2 0 0}$ which increases the original area of the flower bed by $2,100 \mathrm{ft}^{2}$, not $\mathbf{3 5 0} \mathrm{ft}^{2}$.
The border is $\frac{\mathbf{1 0}}{\mathbf{2}}=\mathbf{5 f t} .(10+\mathbf{2 0})(\mathbf{1 0}+\mathbf{5})=450 \mathrm{ft}^{2}$. The increase is $\mathbf{4 5 0}-\mathbf{1 0 0}=\mathbf{3 5 0} \mathrm{ft}^{2}$.
The dimensions of the increased rectangle are $\mathbf{3 0}$ by $\mathbf{1 5} \mathrm{ft}$.

## Chapter 43

## Solving Application Problems by Factoring

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### 43.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqShSBwE1AV1h6Te4-IhhuMM\&feature=view_all

### 43.2 Basics

Remember that you know the answer to every application (aka word) problem. The answer (solution) is $\boldsymbol{x}$.

### 43.3 Examples

Example 1:
Two ships leave harbor at the same time. One travels north, the other travels west. How far did the westbound ship travel when they are 130 miles apart and the northbound ship traveled 20 miles more than twice the westbound ship.

## Solution:

Let $\boldsymbol{x}$ be the distance the westbound ship traveled.


Then $2 \boldsymbol{x}+\mathbf{2 0}$ is the distance traveled by the northbound ship.
By the Pythagorean Theorem

$$
\begin{aligned}
x^{2}+(2 x+20)^{2} & =130^{2} \\
x^{2}+4 x^{2}+80 x+20^{2} & =16,900 \\
5 x^{2}+80 x+400-16,900 & =0 \\
5 x^{2}+80 x-16,500 & =0 \\
5\left(x^{2}+16 x-3,300\right) & =0
\end{aligned}
$$

| $\begin{gathered} P=-3,300 \\ S=16 \end{gathered}$ |  |  |
| :---: | :---: | :---: |
| -1 | 3300 |  |
| -2 | 1650 |  |
| -3 | 1100 |  |
| -4 | 825 |  |
| -5 | 660 |  |
| -10 | 330 |  |
| -15 | 220 |  |
| -20 | 165 |  |
| -25 | 132 |  |
| -30 | 110 |  |
| $-50$ | 66 | sum $-50+66=16$ |
| $x^{2}+$ | $\begin{aligned} & 0= \\ &= \\ &= \end{aligned}$ | $\begin{aligned} & x^{2}-50 x+66 x-3,300 \\ & x(x-50)+66(x-50) \\ & (x-50)(x+66) \end{aligned}$ |

$\boldsymbol{x}=-\mathbf{6 6}$ to be rejected because distances are not negative.
Then $\boldsymbol{x}=\mathbf{5 0}$. The westbound ship traveled 50 miles.

## Example 2:

You stand at the edge of a cliff 24 ft above the ocean. You throw a projectile vertically up at 20 ft per second. The equation giving the height of the projectile $\boldsymbol{t}$ seconds after release is $\boldsymbol{H}(\boldsymbol{t})=-\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}}+\mathbf{2 0} \boldsymbol{t}+\mathbf{2 4}$. How many seconds after release will the projectile hit the water?

## Solution:

The projectile reaches the ocean when $\boldsymbol{H}(\boldsymbol{t})=\mathbf{0}$.

| $-16 t^{2}+20 t+24$ |  | $=$ | 0 |
| ---: | :---: | :--- | :--- |
| $4 t^{2}-5 t-6$ | $=$ | 0 |  |$\quad$ divide by -5

Then $\boldsymbol{t}=-\frac{\mathbf{3}}{\mathbf{4}}$ which is meaningless in the present context
or

## Fold along dashed line

$\boldsymbol{t}=\mathbf{2}$. Thus the projectile will it the water 2 seconds after releases.

## Example 3:

You are to fabricate a box by taking a square piece of cardboard, cutting congruent squares with sides 3 inches from each of the four corners, and folding the sides. Find the measure of the side of the cardboard if the volume is to be 243 cubic inches. The box is open on top.


## Solution:

The base of the box is $(x-6)^{2}=x^{2}-12 x+36$ and the volume is

| $6)^{2}=3\left(x^{2}-12 x+36\right)=243$ in $^{3}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $3\left(x^{2}-12 x+36\right)$ | $=$ | 243 |  |
| $x^{2}-12 x+36$ | $=$ | 81 |  |
| $x^{2}-12 x+36-81$ | = | 0 |  |
| $x^{2}-12 x-45$ | = | 0 |  |
| $P=-45$ |  |  |  |
| $S=-12$ |  |  |  |
| 1 |  | -45 |  |
| 3 | \| | -15 | sum $=-12$ |
| $x^{2}+3 x-15 x-45$ | = | 0 |  |
| $x(x+3)-15(x+3)$ | = | 0 |  |
| $(x+3)(x-15)$ | = | 0 |  |

One solution is $\boldsymbol{x}=\mathbf{- 3}$ which is meaningless as a dimension.
$x=15$ inches is the size of a side.

## Example 4:

A 30 by 20 ft swimming pool was surrounded by a concrete walk way of uniform width. The area of the rectangle around the walk way increased the area around the swimming pool by $600 \mathrm{ft}^{2}$. What is the width of the walkway?

## Solution:

The area of the swimming pool is $(\mathbf{2 0})(\mathbf{3 0})=\mathbf{6 0 0} \mathrm{ft}^{2}$.
The uniform width of the walk way is $\boldsymbol{x} \mathrm{ft}$. The area of the rectangle is $(2 x+20)(2 x+30)=4 x^{2}+100 x+600 \mathrm{ft}^{2}$.

The increase in area due to the walk way is

$$
\begin{aligned}
4 x^{2}+100 x+600-600 & =4 x^{2}+100 x \\
& =4 x\left(x^{2}-25\right)=0
\end{aligned}
$$

The width is $\boldsymbol{x}=\mathbf{0}$ which is impractical or $\boldsymbol{x}=\mathbf{- 5}$ which is not a measurement or $\boldsymbol{x}=\mathbf{5}$.
The uniform width of the walkway is $\mathbf{5} \mathrm{ft}$.

## Example 5:

One pipe can fill a tank by itself in 40 minutes more than a second pipe. If the two pipes are used together, it takes 48 minutes to fill the tank. How long does it take the faster pipe to fill the tank alone?

## Solution:

Let $\boldsymbol{x}$ be the number of minutes it takes the faster pipe to fill the tank by itself.
Then $\boldsymbol{x}+\mathbf{4 0}$ is the number of minutes it takes the slower pipe to fill the tank by itself.
In one minute the faster pipe fills $\frac{\mathbf{1}}{\boldsymbol{x}}$ of the tank.
In one minute the slower pipe fills $\frac{1}{x+40}$ of the tank.
In one minute both pipes fill $\frac{1}{48}$ of the tank.
The fraction of the tank filled by the faster pipe plus the fraction of the tank filled by the slower pipe is the same as the fraction of the tank filled by both pipes together.

$$
\begin{array}{rll}
\frac{1}{x}+\frac{1}{x+40} & = & \frac{1}{48} \\
\frac{48 x(x+40)}{x}+\frac{48 x(x+40)}{x+40} & = & \frac{48 x(x+40)}{48} \\
\frac{48(x+40)+48 x}{48 x+(48)(40)+48 x} & = & x(x+40) \\
96 x+1920 & = & \left.x^{2}+40 x\right) \\
48 & = & x^{2}+40 x \\
x^{2}+40 x-96 x-1920 & & = \\
x^{2}-56 x-1920 & & = \\
& P=-1920 & 0 \\
1 & S=-56 & \\
2 & & -1920 \\
3 & & -960 \\
4 & & -480 \\
5 & & -184 \\
6 & & -320 \\
8 & & -240 \\
10 & & -192 \\
12 & & -160 \\
15 & & -128 \\
20 & & -96 \\
24 & & -80
\end{array}
$$

$x^{2}-56 x-1920=(x-80)(x+24)=0$
We reject $\boldsymbol{x}=\mathbf{- 2 4}$ because time is not a negative number.
It takes the faster pipe 80 minutes to fill the tank by itself.

## Example 6:

Two rectangular corrals are constructed side by side as shown. The base of the outside rectangle is 10 ft more than its width. The total area of the two orals is $3,000 \mathrm{ft}^{2}$.
What is the cost of fencing if each ft of fencing costs $\mathbf{\$ 1 0}$ ?

## Solution:

The area is

$$
\begin{aligned}
x(x+10) & =3000 \\
x^{2}+10 x-3000 & =0 \\
(x+60)(x-50) & =0
\end{aligned}
$$

Only $\boldsymbol{x}=\mathbf{5 0}$ makes sense as a dimension.


The total fence measures $\mathbf{3 x + 2}(\boldsymbol{x}+\mathbf{1 0})=\mathbf{5 x}+\mathbf{2 0} \mathrm{ft}$. which costs $\mathbf{1 0} \frac{\text { dollars }}{\mathrm{ft}} \cdot[5(50)+\mathbf{2 0}] \mathrm{ft}$ or $10(270)=\$ 2,700$

### 43.4 Exercise 43

1. Two ships leave harbor at the same time. One travels north, the other travels west. How far did the westbound ship travel when they are 125 miles apart and the northbound ship traveled 15 miles more than three times the westbound ship.

2. You stand at the edge of a pier 30 ft above the ocean. You throw a stone vertically up at 28 ft per second. The equation giving the height of the projectile $\boldsymbol{t}$ seconds after release is $\boldsymbol{H}(\boldsymbol{t})=-\mathbf{1 6} \boldsymbol{t}^{2}+\mathbf{2 8 t}+\mathbf{3 0}$. How many seconds after release will the projectile hit the water?
3. You are to fabricate a box by taking a square piece of cardboard, cutting congruent squares with sides 4 inches from each of the four corners, and folding the sides. Find the measure of the side of the cardboard if the volume is to be 484 cubic inches.
The box is open on top.
4. A 40 by 20 ft swimming pool was surrounded by a concrete walk way of uniform width. The area of the rectangle around the walk way increased the area around the swimming pool by $544 \mathrm{ft}^{2}$. What is the width of the walk way?

5. One pipe can fill a tank by itself in 50 minutes more than a second pipe. If the two pipes are used together, it takes 60 minutes to fill the tank. How long does it take the faster pipe to fill the tank alone?
6. Two rectangular corrals are constructed side by side as shown. The base of the outside rectangle is 20 ft more than twice its width. The total area of the two orals is $4,000 \mathrm{ft}^{2}$.
What is the cost of fencing if each ft of fencing costs $\mathbf{\$ 1 2}$ ?


## STOP!

1. Two ships leave harbor at the same time. One travels north, the other travels west. How far did the westbound ship travel when they are 125 miles apart and the northbound ship traveled 15 miles more than three times the westbound ship.

## Solution:

Let $\boldsymbol{x}$ be the distance the westbound ship traveled.
Then $\mathbf{2 x}+\mathbf{2 0}$ is the distance traveled by the northbound ship.
By the Pythagorean Theorem

$$
\begin{aligned}
x^{2}+(3 x+15)^{2} & =125^{2} \\
x^{2}+9 x^{2}+90 x+15^{2} & =15,625 \\
5 x^{2}+80 x+900-15,625 & =0 \\
5 x^{2}+80 x-14,625 & =0 \\
5\left(x^{2}+20 x-2,925\right) & =0
\end{aligned}
$$



| $P=-2,925$ |  |
| :--- | :--- |
| $S=20$ |  |
| -1 | 2925 |
| -3 |  |
| -5 | 975 |
| -9 |  |
| -25 | 325 |
| -45 |  |

$$
\begin{aligned}
x^{2}+20 x-2,925 & =x^{2}-45 x+65 x-2,925 \\
& =x(x-45)+65(x-45) \\
& =(x-45)(x+65)
\end{aligned}
$$

$\boldsymbol{x}=-\mathbf{6 5}$ to be rejected because distances are not negative.
Then $\boldsymbol{x}=\mathbf{4 5}$. The westbound ship traveled 45 miles.
2. You stand at the edge of a pier 30 ft above the ocean. You throw a stone vertically up at 28 ft per second. The equation giving the height of the projectile $\boldsymbol{t}$ seconds after release is $\boldsymbol{H}(\boldsymbol{t})=-\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}}+\mathbf{2 8 t}+\mathbf{3 0}$. How many seconds after release will the projectile hit the water?

## Solution:

The projectile reaches the ocean when $\boldsymbol{H}(\boldsymbol{t})=\mathbf{0}$.

| $-16 t^{2}+28 t+30$ |  | $=$ |
| ---: | :---: | :--- |
| $8 t^{2}-14 t-15$ | $=$ | 0 |
|  | $P=-120$ |  |
|  | $S=-14$ |  |
| 1 |  | -120 |
| 2 |  | -60 |
| 3 |  | -40 |
| 4 |  | -30 |
| 5 |  | -24 |
| 6 |  | -20 |
| $8 t^{2}+6 t-20 t-15$ | $=$ | 0 |
| $2 t(4 t+3)-5(4 t+3)$ | $=$ | 0 |
| $(4 t+3)(2 t-5)$ | $=$ | 0 |

Then $\boldsymbol{t}=-\frac{\mathbf{3}}{\mathbf{4}}$ which is meaningless in the present context
or
$\boldsymbol{t}=\frac{\mathbf{5}}{\mathbf{2}}$. Thus the stone will it the water $\boldsymbol{t}=\frac{\mathbf{5}}{\mathbf{2}}$ seconds after releases.


The base of the box is $(\boldsymbol{x}-8)^{2}=x^{2}-\mathbf{1 6 x}+\mathbf{6 4}$ and the volume is

$$
4(x-8)^{2}=4\left(x^{2}-16 x+64\right)=484 \text { in }^{3}
$$

| $4\left(x^{2}-16 x+64\right)$ | $=$ | 484 |
| ---: | :--- | :--- |
| $x^{2}-16 x+64$ | $=$ | 121 |
| $x^{2}-16 x+64-121$ | $=$ | 0 |
| $x^{2}-16 x-57$ | $=$ | 0 |
| $r$ | $=-57$ |  |
| $r$ | $=-16$ |  |
| 3 |  |  |
| 3 |  | -57 |
|  | $=-19$ | sum $=-16$ |
| $x^{2}+3 x-19 x-57$ | $=$ | 0 |
| $x(x+3)-19(x+3)$ | $=$ | 0 |
| $(x+3)(x-19)$ | $=$ | 0 |

One solution is $\boldsymbol{x}=\mathbf{- 3}$ which is meaningless as a dimension.
$\boldsymbol{x}=19$ inches is the size of a side.
4. A 40 by 20 ft swimming pool was surrounded by a concrete walk way of uniform width. The area of the rectangle around the walk way increased the area around the swimming pool by $544 \mathrm{ft}^{2}$. What is the width of the walkway?

## Solution:

The area of the swimming pool is $(\mathbf{2 0})(40)=800 \mathrm{ft}^{2}$.
The uniform width of the walk way is $\boldsymbol{x} \mathrm{ft}$. The area of the rectangle is $(2 x+20)(2 x+40)=4 x^{2}+120 x+800 \mathrm{ft}^{2}$.
The increase in area due to the walk way is

$$
\begin{array}{rcl}
4 x^{2}+120 x+800-800 & = & 544 \\
4 x^{2}+120 x+544 & = & 544 \\
4 x^{2}+120 x-544 & = & 0 \\
4\left(x^{2}+30 x-136\right) & = & 0 \\
x^{2}+30 x-136 & = & 0 \\
& P=-136 & \\
& S=30 & \\
& 1 & \\
2 & & -136 \\
& & -68 \\
(x-4)(x+34)=0 & & -34 \\
& &
\end{array}
$$

The width is $\boldsymbol{x}=\mathbf{- 3 4}$ which is not a measurement or $\boldsymbol{x}=\mathbf{4}$.
The uniform width of the walkway is 4 ft .
5. One pipe can fill a tank by itself in 50 minutes more than a second pipe. If the two pipes are used together, it takes 60 minutes to fill the tank. How long does it take the faster pipe to fill the tank alone?

## Solution:

Let $\boldsymbol{x}$ be the number of minutes it takes the faster pipe to fill the tank by itself.
Then $\boldsymbol{x}+\mathbf{5 0}$ is the number of minutes it takes the slower pipe to fill the tank by itself.
In one minute the faster pipe fills $\frac{\mathbf{1}}{\boldsymbol{x}}$ of the tank.
In one minute the slower pipe fills $\frac{1}{x+40}$ of the tank.

In one minute both pipes fill $\frac{\mathbf{1}}{\mathbf{6 0}}$ of the tank.
The fraction of the tank filled by the faster pipe plus the fraction of the tank filled by the slower pipe is the same as the fraction of the tank filled by both pipes together.

$$
\begin{array}{rll}
\begin{aligned}
\frac{1}{x}+\frac{1}{x+50} \\
60 x(x+50)
\end{aligned} & = & \frac{1}{60} \\
\frac{60 x(x+50)}{x}+\frac{1}{x+50} & = & \frac{60 x(x+50)}{60} \\
60(x+50)+60 x & = & x(x+50) \\
60 x+(60)(50)+60 x & = & \left.x^{2}+50 x\right) \\
120 x+3000 & = & x^{2}+50 x \\
x^{2}+50 x-120 x-3000 & = & 0 \\
x^{2}-70 x-3000 & = & 0 \\
& P=-3000 & \\
& S=-70 & \\
1 & & -3000 \\
2 & & -1500 \\
3 & & -1000 \\
4 & & -750 \\
5 & & -600 \\
6 & & -500 \\
8 & & -375 \\
10 & & -300 \\
12 & & -250 \\
15 & & -200 \\
20 & & -150 \\
24 & & -125 \\
25 & & -120 \\
30 & & -100
\end{array}
$$

$x^{2}-70 x-3000=(x+30)(x-100)=0$
We reject $\boldsymbol{x}=\mathbf{- 3 0}$ because time is not a negative number.
It takes the faster pipe 100 minutes to fill the tank by itself.
6. Two rectangular corrals are constructed side by side as shown. The base of the outside rectangle is 20 ft more than twice its width. The total area of the two orals is $4,000 \mathrm{ft}^{2}$.
What is the cost of fencing if each ft of fencing costs $\mathbf{\$ 1 2}$ ?

## Solution:

The area is

$$
\begin{aligned}
x(2 x+20) & =4000 \\
2 x^{2}+20 x-4000 & =0 \\
x^{2}+10 x-2000 & =0 \\
(x+50)(x-40) & =0
\end{aligned}
$$

Only $\boldsymbol{x}=40$ makes sense as a dimension.


The total fence measures $\mathbf{3 x + 2 ( 2 x + 2 0 )}=\mathbf{7 x}+\mathbf{4 0} \mathrm{ft}$. which costs $\mathbf{1 2} \frac{\text { dollars }}{\mathrm{ft}} \cdot[\mathbf{7}(\mathbf{4 0})+\mathbf{2 0}] \mathrm{ft}$ or $10(270)=\$ 3,600$

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## Chapter 44

## Rational Expressions

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### 44.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqTCj38pN76tdzkoagmbFa_6\&feature=view_all

### 44.2 Basics

A rational expression is a ratio (fraction) of two polynomials.
Some ratios can be simplified, added to and/or subtracted from each other. Fractions can be multiplied or divided by each other.

A fraction cannot have a factor of 0 in the denominator.
Remember that $12 \div 4=\frac{12}{4}=\mathbf{3}$ because $4 \cdot \mathbf{3}=12$
numerator $\div$ denominator $=\frac{\text { numerator }}{\text { denominator }}=$ quotient because denominator $\cdot$ quotient $=$ numerator
numerator $\div \mathbf{0}=\frac{\text { numerator }}{\mathbf{0}}=$ quotient
because $\mathbf{0} \cdot$ quotient $=$ numerator or $\mathbf{0}=$ numerator which is (usually) incorrect. We say that division by 0 is undefined.

## Warning:

You will not be able to perform in problems involving rational expressions from now on unless you are proficient in factoring!

### 44.3 Examples

Example 1:
For what value(s) of $x$ is $\frac{(x+4)(2 x-5)}{(x-3)(6 x+7)}$ undefined?

## Solution:

The denominator is 0 if $\boldsymbol{x}=\mathbf{3}$ and $\boldsymbol{x}=-\frac{\mathbf{7}}{\mathbf{6}}$.
Example 2:
Simplify $\frac{4 x^{2}-24 x+27}{108 x^{3}+324 x^{2}-729 x}$

## Solution:

$$
\begin{aligned}
& \frac{4 x^{2}-24 x+27}{108 x^{3}+324 x^{2}-729 x}=\frac{N_{1}}{D_{1}}
\end{aligned}
$$

$$
\begin{aligned}
& N_{1}=4 x^{2}-24 x+27 \quad \text { GCF }=1 \\
& =4 x^{2}-6 x-18 x+27 \\
& =2 x\left(2 x^{2}-3\right)-9(2 x-3) \\
& =(2 x-3)(2 x-9) \\
& D_{1}=108 x^{3}+324 x^{2}-729 x \quad \text { GCF }=27 x \\
& =27 x\left(4 x^{2}+12 x-27\right) \\
& \begin{array}{lcl}
D_{1} & & 27 x\left(4 x^{2}+12 x-27\right) \\
\hline-1 & & 108 \\
-2 & & 54 \\
-3 & & 36 \\
-4 & 27 & \\
-6 & & 18
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
D_{1} & =27 x\left(4 x^{2}-6 x+18 x-27\right) \\
& =27 x[2 x(2 x-3)+9(2 x-3)] \\
& =27 x(2 x-3)(2 x+9) \\
\frac{N_{1}}{D_{1}} & =\frac{(2 x-3)(2 x-9)}{(27 \bar{x})(2 x-3)(2 x+9)}=\frac{2 x-9}{27 x(2 x+9)}
\end{aligned}
$$

Example 3:
Simplify $\frac{2 x^{2}-162}{35 x+21} \cdot \frac{25 x^{2}+30 x+9}{2 x+18}$

## Solution:

$\frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D_{2}}$
$N_{1}=2 x^{2}-162=2\left(x^{2}-81\right)=2(x+9)(x-9)$
$D_{1}=35 x+21=7(5 x+3)$
$N_{2}=25 x^{2}+30 x+9=(5 x+3)^{2}$
$D_{2}=2 x+18=2(x+9)$
$\frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D_{2}}=\frac{2(x+9)(x-9)}{7(5 x+3)} \cdot \frac{(5 x+3)^{2}}{2(x+9)}=\frac{(x-9)(5 x+3)}{7}$
Example 4:
Simplify $\frac{x^{2}-6 x+9}{x^{5}} \div \frac{x^{2}-9}{x^{7}}$

## Solution:

$$
\begin{aligned}
\frac{x^{2}-6 x+9}{x^{5}} \div \frac{x^{2}-9}{x^{7}} & =\frac{x^{2}-6 x+9}{x^{5}} \cdot \frac{x^{7}}{x^{2}-9} \\
& =\frac{(x-3)^{2}}{x^{5}} \cdot \frac{x^{7}}{(x+3)(x-3)}=\frac{x^{2}(x-3)}{x+3}
\end{aligned}
$$

### 44.4 Exercise 44

1. For what value(s) of the unknown will $\frac{(x+6.5)(3 x-10)}{(x-3.5)(8 x+0.7)}$ be undefined?
2. Simplify $\frac{54 x^{3}-30 x^{2}-100 x}{\left.135 x^{3}-375 x^{2}+250 x\right)}=\frac{N_{1}}{D_{1}}$
3. Simplify $\frac{3 x^{2}-243}{16 x^{2}+12 x} \cdot \frac{16 x^{2}+24 x+9}{5 x+45}$
4. Simplify $\frac{x^{2}-49}{x^{9}} \div \frac{x^{2}-14 x+49}{x^{18}}$

## STOP!

1. For what value(s) of the unknown will $\frac{(x+6.5)(3 x-10)}{(x-3.5)(8 x+\mathbf{0 . 7})}$ be undefined?

## Solution:

The denominator is 0 if $\boldsymbol{x}=\mathbf{3 . 5}$ and $\boldsymbol{x}=-\frac{\mathbf{0 . 7}}{\mathbf{8}}=-\frac{\mathbf{7}}{\mathbf{8 0}}$.
2. Simplify $\frac{54 x^{3}-30 x^{2}-100 x}{135 x^{3}-375 x^{2}+250 x}=\frac{N_{1}}{D_{1}}$

## Solution:

$$
\begin{aligned}
& \frac{54 x^{3}-30 x^{2}-100 x}{135 x^{3}-375 x^{2}+250 x}=\frac{N_{1}}{D_{1}} \\
& N_{1}=54 x^{3}-30 x^{2}-100 x \text { GCF }=2 x \\
& =2 x\left(27 x^{2}-15 x-50\right)
\end{aligned}
$$

$$
\frac{N_{1}}{D_{1}}=\frac{2 x(9 x+10)(3 x-5)}{5 x(9 x-10)(3 x-5)}=\frac{2(9 x+10)}{5(9 x-10)}
$$

3. Simplify $\frac{3 x^{2}-243}{16 x^{2}+12 x} \cdot \frac{16 x^{2}+24 x+9}{5 x+45}$

## Solution:

$$
\begin{aligned}
& \frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D_{2}} \\
& N_{1}=3 x^{2}-243=3\left(x^{2}-81\right)=3(x+9)(x-9) \\
& D_{1}=16 x^{2}+12 x=4 x(4 x+3) \\
& N_{2}=16 x^{2}+24 x+9=(4 x+3)^{2} \\
& D_{2}=5 x+45=5(x+9) \\
& \frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D_{2}}=\frac{3(x+9)(x-9)}{4 x(4 x+3)} \cdot \frac{(4 x+3)^{2}}{5(x+9)}=\frac{3(x-9)(4 x+3)}{20 x}
\end{aligned}
$$

4. Simplify $\frac{x^{2}-49}{x^{9}} \div \frac{x^{2}-14 x+49}{x^{18}}$

## Solution:

$$
\begin{aligned}
& \left.D_{1}=135 x^{3}-375 x^{2}+250 x\right) \\
& =\quad 5 x\left(27 x^{2}-75 x+50\right) \\
& P=1350 \\
& \\
& D_{1}=5 x\left(27 x^{2}-75 x+50\right) \\
& =5 x\left(27 x^{2}-30 x-45 x+50\right) \\
& =5 x[3 x(9 x-10)-5(9 x-10)] \\
& =5 x(9 x-10)(3 x-5)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{x^{2}-49}{x^{9}} \div \frac{x^{2}-14 x+49}{x^{18}} \\
= & \frac{(x+7)(x-7)}{x^{9}} \div \frac{(x-7)^{2}}{x^{18}} \\
= & \frac{(x+7)(x-7)}{x^{9}} \cdot \frac{x^{18}}{(x-7)^{2}} \\
= & \frac{x^{9}(x+7)}{x-7}
\end{aligned}
$$

## Chapter 45

## Multiplication and Division of Rational Expressions

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### 45.1 Youtube

https://www.youtube.com/playlist?list=PLbEA2z28bkqQ09nVHkgdjMtej-lpMSUAW\&feature=view_all

### 45.2 Basics

A rational expression is a ratio (fraction) of two polynomials.
From arithmetic we recall that

$$
\begin{aligned}
\frac{4}{27} \cdot \frac{63}{96} & =\frac{2 \cdot 2}{3 \cdot 3 \cdot 3} \cdot \frac{3 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} \\
& =\frac{2}{2} \frac{2}{2} \frac{3}{3} \frac{3}{3} \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} \\
& =1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{7}{144}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\frac{a^{2}}{b^{3}} \cdot \frac{b^{2} c}{a^{6} b} & =\frac{a \cdot a}{b \cdot b \cdot b} \cdot \frac{b \cdot b \cdot c}{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot b} \\
& =\frac{a}{a} \frac{a}{a} \frac{b}{b} \frac{b}{b} \frac{c}{a \cdot a \cdot a \cdot a \cdot b \cdot b} \\
& =1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{c}{a^{4} b^{2}}
\end{aligned}
$$

### 45.3 Examples

## Example 1:

Simplify $\frac{2 x^{8}-x^{7}-15 x^{6}}{-5 x^{4}+13 x^{3}+6 x^{2}}$
Solution:

$$
\begin{aligned}
& \frac{2 x^{8}-x^{7}-15 x^{6}}{-5 x^{4}+13 x^{3}+6 x^{2}}=\frac{N_{1}}{D_{1}} \\
& N_{1}=2 x^{8}-x^{7}-15 x^{6} \quad \text { GCF }=x^{6} \\
& =x^{6}\left(2 x^{2}-x-15\right) \\
& P=-30 \\
& S=-1 \\
& \begin{array}{l|l}
1 & -30 \\
2 & -15
\end{array} \\
& \begin{array}{l|l}
3 & -10 \\
5 & -6
\end{array} \\
& -6-\text { sum }=5-6=-1 \\
& x^{6}\left(2 x^{2}-x-15\right)=x^{6}\left(2 x^{2}+5 x-6 x-15\right) \\
& =x^{6}[x(2 x+5)-3(2 x+5)] \\
& =x^{6}(2 x+5)(x-3) \\
& D_{1}=-5 x^{4}+13 x^{3}+6 x^{2} \quad \text { GCF }=x^{2} \\
& =x^{2}\left(-5 x^{2}+13 x+6\right) \\
& P=-30 \\
& S=+13 \\
& -1 \\
& -2 \text { | } \\
& 15 \text { sum }=-2+15=13 \\
& x^{2}\left(-5 x^{2}+13 x+6\right) \\
& =x^{2}\left(-5 x^{2}-2 x+15 x+6\right) \\
& =x^{2}[-x(5 x+2)+3(5 x+2)] \\
& =x^{2}(5 x+2)(-x+3)
\end{aligned}
$$

45.3. EXAMPLES

$$
\begin{aligned}
\frac{N_{1}}{D_{1}} & =\frac{x^{6}(2 x+5)(x-3)}{x^{2}(5 x+2)(-x+3)} \\
& =\frac{x^{6-2}(2 x+5)(x-3)}{(5 x+2)[-(x-3)]} \\
& =-\frac{x^{4}(2 x+5)(x-3)}{(5 x+2)(x-3)}=-\frac{x^{4}(2 x+5)}{5 x+2}
\end{aligned}
$$

Example 2:
Simplify $\frac{x^{2}-4}{x^{2}+16} \cdot \frac{5 x^{2}+80}{2-x}$
Solution:

$$
\begin{aligned}
& \frac{x^{2}-4}{x^{2}+16} \cdot \frac{5 x^{2}+80}{2-x}
\end{aligned}=\frac{(x+2)(x-2)}{x^{2}+16} \cdot \frac{5\left(x^{2}+16\right)}{(-1)(x-2)}
$$

Simplify

$$
\frac{\left(6 x^{2}+25 x+21\right)\left(14 x^{2}+65 x+56\right)}{\left(6 x^{2}+19 x+15\right)(20 x-34 x+6)} \div \frac{\left(12 x^{2}+56 x+49\right)\left(7 x^{2}+29 x+24\right)}{\left(4 x^{2}-9\right)\left(15 x^{2}+22 x-5\right)}
$$

## Solution:

$$
\begin{aligned}
& \frac{\left(6 x^{2}+25 x+21\right)\left(14 x^{2}+65 x+56\right)}{\left(6 x^{2}+19 x+15\right)\left(20 x^{2}-34 x+6\right)} \div \frac{\left(12 x^{2}+56 x+49\right)\left(7 x^{2}+29 x+24\right)}{\left(4 x^{2}-9\right)\left(15 x^{2}+22 x-5\right)} \\
& =\frac{\left(6 x^{2}+25 x+21\right)\left(14 x^{2}+65 x+56\right)}{\left(6 x^{2}+19 x+15\right)\left(20 x^{2}-34 x+6\right)} \cdot \frac{\left(4 x^{2}-9\right)\left(15 x^{2}+22 x-5\right)}{\left(12 x^{2}+56 x+49\right)\left(7 x^{2}+29 x+24\right)} \\
& =\frac{N_{1} N_{2}}{D_{1} D_{2}} \cdot \frac{N_{3} N_{4}}{D_{3} D_{4}} \\
& N_{1}=6 x^{2}+25 x+21 \quad \text { GCF }=1 \\
& P=126 \\
& S=25 \\
& \begin{array}{l|l}
3 & 41
\end{array} \\
& 6 \text { | } 21 \\
& 7 \quad 18 \text { sum }=7+18=25
\end{aligned}
$$

$$
\begin{aligned}
6 x^{2}+25 x+21 & =6 x^{2}+7 x+18 x+21 \\
& =x(6 x+7)+3(6 x+7) \\
& =(6 x+7)(x+3)
\end{aligned}
$$

$$
\begin{array}{rlr}
N_{3} & =4 x^{2}-9 & \text { GCF }=1 \\
& =(2 x)^{2}-3^{2} & \\
& =(2 x+3)(2 x-3)
\end{array}
$$

$$
N_{4}=15 x^{2}+22 x-5 \mathrm{GCF}=1
$$

$$
P=-75
$$

$$
S=22
$$

$$
-1
$$

$$
-3 \quad \mid \quad 25 \quad \text { sum }=-3+25=22
$$

$$
15 x^{2}+22 x-5=15 x^{2}-3 x+25 x-5
$$

$$
=3 x(5 x-1)+5(5 x-1)
$$

$$
=(5 x-1)(3 x+5)
$$

$$
D_{1}=6 x^{2}+19 x+15 \mathrm{GCF}=1
$$

```
P=90
    S=19
        90
        4 5
        30
        18
        15
        10 sum = 9 + 10 = 19
```

$$
\begin{aligned}
& N_{2}=14 x^{2}+65 x+56 \quad \text { GCF }=1 \\
& P=784 \\
& S=65 \\
& 14 x^{2}+65 x+56=14 x^{2}+16 x+49 x+56 \\
& =2 x(7 x+8)+7(7 x+8) \\
& =(7 x+8)(2 x+7)
\end{aligned}
$$

$$
\begin{aligned}
6 x^{2}+19 x+15 & =6 x^{2}+9 x+10 x+15 \\
& =3 x(2 x+3)+5(2 x+3) \\
& =(2 x+3)(3 x+5)
\end{aligned}
$$

$$
\begin{aligned}
& D_{2}=20 x^{2}-34 x+6 \quad \text { GCF }=2 \\
& =2\left(10 x^{2}-17 x+3\right)
\end{aligned}
$$

$$
\begin{aligned}
& 2\left(10 x^{2}-17 x+3\right)=2\left(10 x^{2}-2 x-15 x+3\right) \\
& =2[2 x(5 x-1)-3(5 x-1)] \\
& =2(5 x-1)(2 x-3)
\end{aligned}
$$

$$
\begin{aligned}
& D_{3}=12 x^{2}+56 x+49 \text { GCF }=1 \\
& P=588 \\
& S=56 \\
& \begin{array}{rll}
1 & \mid l & 588 \\
2 & & 294 \\
4 & & 147 \\
7 & & 84 \\
14 & & 42 \text { sum }=14+42=56
\end{array} \\
& 12 x^{2}+56 x+49=12 x^{2}+14 x+42 x+49 \\
& =2 x(6 x+7)+7(6 x+7) \\
& =(6 x+7)(2 x+7)
\end{aligned}
$$

$$
\begin{aligned}
& D_{4}=7 x^{2}+29 x+24 \text { GCF }=1 \\
& P=168 \\
& S=29 \\
& \text { | } 168 \\
& 84 \\
& 56 \\
& 42 \\
& 28 \\
& 24 \\
& 21 \text { sum }=8+21=29 \\
& 7 x^{2}+29 x+24=7 x^{2}+8 x+21 x+24 \\
& =x(7 x+8)+3(7 x+8) \\
& =(7 x+8)(x+3)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{N_{1} N_{2}}{D_{1} D_{2}} \cdot \frac{N_{3} N_{4}}{D_{3} D_{4}} \\
= & \frac{(6 x+7)(x+3)(7 x+8)(2 x+7)}{(2 x+3)(3 x+5)(5 x-1)(2)(2 x-3)} \cdot \frac{(2 x+3)(2 x-3)(5 x-1)(3 x+5)}{(6 x+7)(2 x+7)(7 x+8)(x+3)} \\
= & \frac{(x+3)(7 x+8)(2 x+7)}{(3 x+5)(5 x-1)(2)(2 x-3)} \cdot \frac{(2 x-3)(5 x-1)(3 x+5)}{(2 x+7)(7 x+8)(x+3)} \\
= & \frac{(7 x+8)(2 x+7)}{(5 x-1)(2)(2 x-3)} \cdot \frac{(2 x-3)(5 x-1)}{(2 x+7)(7 x+8)} \\
= & \frac{(2 x+7)}{(2)(2 x-3)} \cdot \frac{(2 x-3)}{(2 x+7)}=\frac{1}{2}
\end{aligned}
$$

### 45.4 Exercise 44

1. Simplify $\frac{15 x^{2}-17 x-4}{21 x^{7}-22 x^{6}+8 x^{5}}$
2. Simplify $\frac{x^{3}-64 x}{x^{2}-8 x+16} \cdot \frac{4-x}{5 x^{5}}$
3. Simplify $\frac{4 x^{2}+4 x+1}{4 x^{2}+17 x+15} \div \frac{8 x^{3}-2 x}{8 x^{2}+6 x-5}$

## $S T O P!$

1. Simplify $\frac{15 x^{2}-17 x-4}{21 x^{7}-22 x^{6}+8 x^{5}}$

## Solution:

$$
\begin{aligned}
& \frac{15 x^{2}-17 x-4}{21 x^{7}-34 x^{6}+8 x^{5}}=\frac{N_{1}}{D_{1}} \\
& N_{1}=15 x^{2}-17 x-4 \text { GCF }=1 \\
& P=-60 \\
& S=-17 \\
& \begin{array}{l|l}
1 & -60 \\
2 & -30
\end{array} \\
& 3 \quad \mid \quad-20 \quad \text { sum }=3-20=-17 \\
& 15 x^{2}-17 x-4=15 x^{2}+3 x-20 x-4 \\
& =3 x(5 x+1)-4(5 x+1) \\
& =(5 x+1)(3 x-4)
\end{aligned}
$$

$$
\begin{aligned}
& D_{1}=21 x^{7}-34 x^{6}+8 x^{5} \quad \text { GCF }=x^{5} \\
& =x^{5}\left(21 x^{2}-34 x+8\right) \\
& P=168 \\
& S=-34 \\
& \begin{array}{l|l}
-1 & -168 \\
-2 & -84 \\
-3 & -56 \\
-4 & -42 \\
-6 & \\
\hline
\end{array} \\
& x^{5}\left(21 x^{2}-34 x+8=x^{5}\left(21 x^{2}-6 x-28 x+8\right)\right. \\
& =x^{5}[3 x(7 x-2)-4(7 x-2) \\
& =x^{5}(7 x-2)(3 x-4) \\
& \frac{N_{1}}{D_{1}}=\frac{(5 x+1)(3 x-4)}{x^{5}(7 x-2)(3 x-4)}=\frac{5 x+1}{x^{5}(7 x-2)}
\end{aligned}
$$

2. Simplify $\frac{x^{3}-64 x}{x^{2}-8 x+16} \cdot \frac{4-x}{5 x^{5}}$

## Solution:

$\frac{N_{1} \cdot N_{2}}{D_{1} \cdot D_{2}}=\frac{x^{3}-64 x}{x^{2}-8 x+16} \cdot \frac{4-x}{5 x^{5}}$
$N_{1}=x^{3}-64 x=x\left(x^{2}-64\right)=x(x+8)(x-8)$
$N_{2}=4-x=(-1)(4-x)$
$D_{1}=x^{2}-8 x+16=(x-4)^{2}$
because $a^{2}-2 a b+b^{2}=(a-b)^{2}$
$D_{2}=5 x^{5}$

$$
\begin{aligned}
\frac{N_{1} \cdot N_{2}}{D_{1} \cdot D_{2}} & =\frac{x^{3}-64 x}{x^{2}-8 x+16} \cdot \frac{4-x}{5 x^{5}} \\
& =\frac{x(x+8)(x-8)}{(x-4)^{2}} \cdot \frac{(-1)(x-4)}{5 x^{5}} \\
& =\frac{(x+8)(x-8)}{(x-4)} \cdot \frac{(-1)}{5 x^{4}} \\
& =-\frac{(x+8)(x-8)}{5 x^{4}(x-4)}
\end{aligned}
$$

3. Simplify $\frac{4 x^{2}+4 x+1}{4 x^{2}+17 x+15} \div \frac{8 x^{3}-2 x}{8 x^{2}+6 x-5}$

## Solution:

$$
\begin{aligned}
& \frac{4 x^{2}+4 x+1}{4 x^{2}+17 x+15} \div \frac{8 x^{3}-2 x}{8 x^{2}+6 x-5} \\
& =\frac{4 x^{2}+4 x+1}{4 x^{2}+17 x+15} \cdot \frac{8 x^{2}+6 x-5}{8 x^{3}-2 x}=\frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D 2} \\
& N_{1}=4 x^{2}+4 x+1 \quad \mathrm{GCF}=1 \\
& =(2 x)^{2}+2(2 x)(1)+(1)^{2} \\
& =(2 x+1)^{2} \quad a^{2}+2 a b+b^{2}=(a+b)^{2} \\
& D_{1}=4 x^{2}+17 x+15 \quad \text { GCF }=1 \\
& P=60 \\
& S=17 \\
& 12 \text { sum }=5+12=17 \\
& 4 x^{2}+17 x+15=4 x^{2}+5 x+12 x+15 \\
& =x(4 x+5)+3(4 x+5) \\
& =(4 x+5)(x+3) \\
& N_{2}=8 x^{2}+6 x-5 \quad \text { GCF }=2 \\
& P=-40 \\
& S=6 \\
& \begin{array}{l|l}
-1 & \begin{array}{l}
40 \\
-2
\end{array} \\
-4 & \begin{array}{l}
20 \\
10
\end{array} \quad \text { sum }=-4+10=6
\end{array} \\
& 8 x^{2}+6 x-5=8 x^{2}-4 x+10 x-5 \\
& =4 x(2 x-1)+5(2 x-1) \\
& =(2 x-1)(4 x+5) \\
& D_{2}=8 x^{3}-2 x \quad \text { GCF }=2 x \\
& =2 x\left(4 x^{2}-1\right) \\
& =2 x(2 x+1)(2 x-1) \\
& =\frac{N_{1}}{D_{1}} \cdot \frac{N_{2}}{D 2} \\
& =\frac{(2 x+1)^{2}}{(4 x+5)(x+3)} \cdot \frac{(2 x-1)(4 x+5)}{2 x(2 x+1)(2 x-1)} \\
& =\frac{2 x+1}{2 x(x+3)}
\end{aligned}
$$

## Chapter 46

# Addition and Subtraction of Rational Expressions with Like Denominators 

(c) H. Feiner 2011

### 46.1 Youtube

https://www.youtube.com/playlist?list=PL83E1A391523F234C\&feature=view_all

### 46.2 Basics

Let's review the concept of fraction and assume that the whole is a regular piece of paper $\mathbf{8}$ by $\mathbf{1 2}$ inches (rounded for easier numbers).

A fraction is a part of a whole. Let the whole be subdivided into three (identical) parts.
Now add one part to another part to get one piece with two parts. Thus
$\frac{1}{3}+\frac{1}{3}=\frac{1+1}{3}=\frac{2}{3}$.


Again add another part to get three parts. Thus
$\frac{\mathbf{2}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{3}}=\frac{\mathbf{2}+\mathbf{1}}{\mathbf{3}}=\frac{\mathbf{3}}{\mathbf{3}}=1$, that is the whole piece of paper.
Can we keep on adding parts? Yes indeed.
$\frac{\mathbf{3}}{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{3}}=\frac{\mathbf{3}+\mathbf{1}}{\mathbf{3}}=\frac{\mathbf{4}}{\mathbf{3}}=1+\frac{1}{3}$ written without the $+\operatorname{sign} 1 \frac{1}{3}$.
Note that the denominators of all fractional parts are the same. This is crucial when adding and/or subtracting fractions. If the denominators were not the same, then the fraction parts would be of different sizes. We would be adding apples to oranges. We'll deal with that situation later.

### 46.3 Examples

Example 1:
Add $\frac{\mathbf{2}}{\mathbf{7}}+\frac{\mathbf{3}}{\mathbf{7}}$.

## Solution:

$\frac{2}{7}+\frac{3}{7}=\frac{2+3}{7}=\frac{5}{7}$
Example 2:
Add $\frac{\mathbf{2}}{\boldsymbol{a}}+\frac{\mathbf{3}}{\boldsymbol{a}}$.

## Solution:

$\frac{2}{a}+\frac{3}{a}=\frac{2+3}{a}=\frac{5}{a}$
Example 3:

Add $\frac{\boldsymbol{a}}{\boldsymbol{x}}+\frac{\boldsymbol{b}}{\boldsymbol{x}}$.

## Solution:

$\frac{a}{x}+\frac{b}{x}=\frac{a+b}{x}$.
Note that you add numerators, not denominators.
Example 4:
Add $\frac{2}{x-3}+\frac{3}{3-x}$.
Solution:
$\frac{2}{x-3}+\frac{3}{3-x}=\frac{2}{x-3}-\frac{3}{x-3}=\frac{2-3}{x-3}=-\frac{1}{x-3}$

Example 5:
Subtract $\frac{3 x+1}{2 x-3}-\frac{x+4}{2 x-3}$.

## Solution:

$$
\begin{aligned}
\frac{3 x+1}{2 x-3}-\frac{x+4}{2 x-3} & =\frac{3 x+1-(x+4)}{2 x-3} \\
& \text { note parentheses around } x+4 \\
& =\frac{3 x+1-x-4}{2 x-3} \\
& =\frac{2 x-3}{2 x-3}=1
\end{aligned}
$$

Example 6:
Subtract $\frac{x^{2}-x-6}{x^{2}-3 x-10}-\frac{2 x^{2}-25 x+63}{2 x^{2}-17 x+35}$.

## Solution:

$$
\frac{x^{2}-x-6}{x^{2}-3 x-10}-\frac{2 x^{2}-25 x+63}{2 x^{2}-17 x+35}=\frac{N_{1}}{D_{1}}-\frac{N_{2}}{D_{2}}
$$

What is this? The denominators are not the same. Can we reduce the fractions before subtracting and then happen to get the same denominators?

$$
\begin{aligned}
& N_{1}=x^{2}-x-6 \quad \text { GCF }=1 \\
& =(x-3)(x+2) \\
& D_{1}=x^{2}-3 x-10 \quad \mathrm{GCF}=1 \\
& =(x-5)(x+2) \\
& N_{2}=2 x^{2}-25 x+63 \mathrm{GCF}=1 \\
& P=126 \\
& S=-25 \\
& \quad \text { sum }=-7-18=-25 \\
& 2 x^{2}-25 x+63=2 x^{2}-7 x-18 x+63 \\
& =x(2 x-7)-9(2 x-7) \\
& =(2 x-7)(x-9)
\end{aligned}
$$

$$
\begin{aligned}
& D_{2}=2 x^{2}-17 x+35 \mathrm{GCF}=1 \\
& P=70 \\
& \\
& 2 x^{2}-17 x+35=2 x^{2}-7 x-10 x+35 \\
& =x(2 x-7)-5(2 x-7) \\
& =(2 x-7)(x-5)
\end{aligned}
$$

$$
\begin{aligned}
\frac{N_{1}}{D_{1}}-\frac{N_{2}}{D_{2}} & =\frac{(x-3)(x+2)}{(x-5)(x+2)}-\frac{(2 x-7)(x-9)}{(2 x-7)(x-5)} \\
& =\frac{x-3}{x-5}-\frac{x-9}{x-5} \\
& =\frac{x-3-(x-9)}{x-5} \\
& =\frac{x-3-x+9}{x-5} \\
& =\frac{6}{x-5}
\end{aligned}
$$

### 46.4 Exercise 46

1. $\operatorname{Add} \frac{\mathbf{5}}{\mathbf{9}}+\frac{8}{\mathbf{9}}$.
2. Add $\frac{\mathbf{5}}{\boldsymbol{m}}+\frac{\mathbf{8}}{\boldsymbol{m}}$.
3. $\operatorname{Add} \frac{\boldsymbol{x}}{\boldsymbol{z}}+\frac{\boldsymbol{y}}{\boldsymbol{z}}$.
4. Add $\frac{\mathbf{9}}{\boldsymbol{a}-\mathbf{5}}+\frac{\mathbf{7}}{\mathbf{5}-\boldsymbol{a}}$.
5. Add $\frac{8 x-2}{3 x-4}+\frac{2 x+6}{4-3 x}$.
6. Subtract $\frac{x^{2}-x-6}{x^{2}-3 x-10}-\frac{2 x^{2}-25 x+63}{2 x^{2}-17 x+35}$.

## STOP!

1. $\operatorname{Add} \frac{\mathbf{5}}{\mathbf{9}}+\frac{\mathbf{8}}{\mathbf{9}}$.

## Solution:

$\frac{5}{9}+\frac{8}{9}=\frac{5+8}{9}=\frac{13}{9}=1 \frac{4}{9}$
2. Add $\frac{\mathbf{5}}{\boldsymbol{m}}+\frac{8}{\boldsymbol{m}}$.

## Solution:

$\frac{5}{m}+\frac{8}{m}=\frac{5+8}{m}=\frac{13}{m}$
3. Add $\frac{\boldsymbol{x}}{\boldsymbol{z}}+\frac{\boldsymbol{y}}{\boldsymbol{z}}$.

Solution:
$\frac{x}{z}+\frac{y}{z}=\frac{x+y}{z}$.
Note that you add numerators, not denominators.
4. Add $\frac{\mathbf{9}}{\boldsymbol{a}-\mathbf{5}}+\frac{\mathbf{7}}{\mathbf{5}-\boldsymbol{a}}$

Solution:

$$
\frac{9}{a-5}+\frac{7}{5-a}=\frac{9}{a-5}-\frac{7}{a-5}=\frac{9-7}{a-5}=\frac{2}{a-5}
$$

5. Add $\frac{8 x-2}{3 x-4}+\frac{2 x+6}{4-3 x}$.

Solution:

$$
\begin{aligned}
\frac{8 x-2}{3 x-4}+\frac{2 x+6}{4-3 x} & =\frac{8 x-2}{3 x-4}-\frac{2 x+6}{3 x-4} \\
& =\frac{8 x-2-(2 x+6)}{3 x-4} \quad \text { note parentheses around } 2 x+6 \\
& =\frac{8 x-2-2 x-6}{3 x-4} \\
& =\frac{6 x-8}{3 x-4} \\
& =\frac{2(3 x-4)}{3 x-4}=2
\end{aligned}
$$

6. Subtract $\frac{x^{2}-x-6}{x^{2}-3 x-10}-\frac{2 x^{2}-25 x+63}{2 x^{2}-17 x+35}$.

## Solution:

$$
\frac{x^{2}-3 x-18}{2 x^{2}-13 x+6}-\frac{3 x^{2}+19 x-40}{6 x^{2}-13 x+5}=\frac{N_{1}}{D_{1}}-\frac{N_{2}}{D_{2}}
$$

What is this? The denominators are not the same. Can we reduce the fractions before subtracting and then happen to get the same denominators?

$$
N_{2}=3 x^{2}+19 x-40 \quad \mathrm{GCF}=1
$$

$$
\begin{aligned}
& N_{1}=x^{2}-3 x-18 \quad \text { GCF }=1 \\
& =(x-6)(x+3) \\
& D_{1}=2 x^{2}-13 x+6 \text { GCF }=1 \\
& P=12 \\
& S=-13 \\
& -1 \quad \mid \quad-12 \quad \text { sum }=-1-12=-13 \\
& 2 x^{2}-13 x+6=2 x^{2}-x-12 x+6 \\
& =x(2 x-1)-6(2 x-1) \\
& =(2 x-1)(x-6)
\end{aligned}
$$

$$
\begin{aligned}
& D_{2}=6 x^{2}-13 x+5 \quad \mathrm{GCF}=1 \\
& P=30 \\
& \begin{array}{c|l}
S=-13 \\
-1 & -30 \\
-2 & -15 \\
-3 & -10 \quad \text { sum }=-3-10=-13
\end{array} \\
& \begin{aligned}
6 x^{2}-13 x+5 & =6 x^{2}-3 x-10 x+5 \\
& =3 x(2 x-1)-5(2 x-1) \\
& =(2 x-1)(3 x-5)
\end{aligned}
\end{aligned}
$$

$$
\frac{N_{1}}{D_{1}}-\frac{N_{2}}{D_{2}}=\frac{(x-6)(x+3)}{(2 x-1)(x-6)}-\frac{(3 x-5)(x+8)}{(2 x-1)(3 x-5)}
$$

$$
=\frac{x+3}{2 x-1}-\frac{x+8}{2 x-1}
$$

$$
=\frac{x+3-(x+8)}{2 x-1}
$$

$$
=\frac{x+3-x-8}{2 x-1}
$$

$$
=\frac{-5}{2 x-1}
$$

$$
\begin{aligned}
& P=-120 \\
& S=19 \\
& \begin{array}{l|l}
-1 & 120 \\
-2 & 60 \\
-3 & 40 \\
-4 & 30 \\
-5 & 24
\end{array} \\
& \text { sum }=-5-24=-19 \\
& 3 x^{2}+19 x-40=3 x^{2}-5 x+24 x-40 \\
& =x(3 x-5)+8(3 x-5) \\
& =(3 x-5)(x+8)
\end{aligned}
$$

Blank page for student note taking.

## Chapter 47

## Least Common Denominator of Rational Expressions

(c) H. Feiner 2011

### 47.1 Youtube

https://www.youtube.com/playlist?list=PLCE595D84681E81FE\&feature=view_all

### 47.2 Basics

We saw that adding and subtracting fractions required the same denominator. If we use the Least Common Denominator (Least Common Multiple - LCM - of all the denominators), then we work with the smallest possible numbers (expressions).

Let's review the LCM of $\mathbf{1 2 0}, \mathbf{5 0 4}$, and $\mathbf{4 3 2}$
First get prime factorizations.


| 120 | 2 | divide 120 by 2 to get $\mathbf{6 0}$ |
| ---: | :--- | :--- |
| 60 | 2 |  |
| 30 | 2 |  |
| 15 | 3 |  |
| 5 | 5 |  |
| 1 |  |  |

Thus $120=2^{3} \cdot 3 \cdot 5$.

| 504 | 2 | divide $\mathbf{5 0 4}$ by 2 to get $\mathbf{2 5 2}$ |
| ---: | :--- | :--- |
| 252 | 2 |  |
| 126 | 2 |  |
| 63 | 3 |  |
| 21 | 3 |  |
| 7 | 7 |  |
| 1 |  |  |

Thus $504=2^{3} \cdot 3^{2} \cdot 7$

| 432 | 2 | divide 432 by 2 to get $\mathbf{2 1 6}$ |
| ---: | :--- | :--- |
| 216 | 2 |  |
| 108 | 2 |  |
| 54 | 2 |  |
| 27 | 3 |  |
| 9 | 3 |  |
| 3 | 3 |  |
| 1 |  |  |

Thus $\mathbf{4 3 2}=\mathbf{2}^{4} \cdot \mathbf{3}^{3}$
Next put the prime factors in columns. Place as many like factors in the same column as possible. Do not mix unlike factors in the same column. We do not use exponents because of remembering use of lowest or highest exponent.


Advantages: Suppose 120, 504, and $\mathbf{4 3 2}$ are denominators. What do you need to multiply 120 by to get the LCM (LCD)? Answer: $2 \cdot \mathbf{3} \cdot \mathbf{3} \cdot \mathbf{7}=\mathbf{1 2 6}$, the missing factors in the horizontal line of $\mathbf{1 2 0}$ in the previous array. (You can use a calculator with numbers, but not with letters.) Learn my way!

Similarly what do you need to multiply 504 by to get the LCM? Answer: $\mathbf{2} \cdot \mathbf{3} \cdot \mathbf{5}=\mathbf{3 0}$, the missing factors in the horizontal line of $\mathbf{5 0 4}$ in the previous array.

What do you need to multiply 432 by to get the LCM? Answer: 5•7=35, the missing factors in the horizontal line of 504 in the previous array.

### 47.3 Examples

## Example 1:

Find the LCM of
$20 x^{2}+40 x-60$,
$10 x^{3}-25 x^{2}+15 x$, and
$16 x^{4}-56 x^{3}+24 x^{2}$.

## Solution:

$$
\begin{aligned}
20 x^{2}+40 x-60 & =20\left(x^{2}+2 x-3\right) \\
& =20(x-1)(x+3) \\
10 x^{3}-25 x^{2}+15 x & =5 x\left(2 x^{2}-5 x+3\right) \\
& =5 x\left(2 x^{2}-5 x+3\right)
\end{aligned}
$$

$$
\begin{aligned}
& \quad P=6 \\
& S=-5 \\
& -1 \\
& -2
\end{aligned} \quad \begin{aligned}
& \\
& 5 x\left(2 x^{2}-5 x+3\right)=5 x\left(2 x^{2}-2 x-3 x+3\right) \\
&=5 x[2 x(x-1)-3(x-1) \\
&=5 x(x-1)(2 x-3)
\end{aligned}
$$

$$
\begin{aligned}
8 x^{2}\left(2 x^{2}+3 x-9\right) & =8 x^{2}\left(2 x^{2}-3 x+6 x-9\right) \\
& =8 x^{2}[x(2 x-3)+3(2 x-3)] \\
& =8 x^{2}(2 x-3)(x+3)
\end{aligned}
$$

$$
\begin{array}{rlllllllll}
20 x^{2}+40 x-60 & = & 2 & 2 & 5 & & & (x-1) & (x+3) & \\
10 x^{3}-25 x^{2}+15 x & = & & & 5 & x & & (x-1) & & (2 x-3) \\
16 x^{4}+24 x^{3}-72 x^{2} & = & 2 & 2 & 2 & & x & x & & (x+3) \\
\hline \text { LCM } & = & 2 & 2 & 2 & 5 & x & x & (x-1) & (x+3) \\
\hline
\end{array}
$$

What do you multiply $20 x^{2}+40 x-60$ by to get the LCM? Answer: $2 x^{2}(2 x-3)$
What do you multiply $10 x^{\mathbf{3}}-\mathbf{2 5} \boldsymbol{x}^{2}+\mathbf{1 5 x}$ by to get the LCM? Answer: $\mathbf{8 x}(\boldsymbol{x}+3)$
What do you multiply $\mathbf{1 6} \boldsymbol{x}^{\mathbf{4}}+\mathbf{2 4} \boldsymbol{x}^{\mathbf{3}}-\mathbf{7 2} \boldsymbol{x}^{\mathbf{2}}$ by to get the LCM? Answer: $\mathbf{5}(\boldsymbol{x}-\mathbf{1})$

### 47.4 Exercise 47

1. Find the LCM of

$$
\begin{aligned}
& 18 x^{3}+126 x^{2}+216 x \\
& 15 x^{4}-30 x^{3}-15 x^{2}, \text { and } \\
& 42 x^{5}-42 x^{4}-840 x^{3}
\end{aligned}
$$

$$
\begin{aligned}
& 16 x^{4}+24 x^{3}-72 x^{2}=8 x^{2}\left(2 x^{2}+3 x-9\right) \\
& P=-18 \\
& S=3 \\
& \begin{array}{l|l}
-1 & 18 \\
-2 & 9 \\
-3 & 6
\end{array} \quad \text { sum }=-3+6=3
\end{aligned}
$$

## STOP!

1. Find the LCM of
$18 x^{3}+126 x^{2}+216 x$,
$15 x^{4}-30 x^{3}-15 x^{2}$, and
$42 x^{5}-42 x^{4}-840 x^{3}$.

## Solution:

$$
\begin{aligned}
18 x^{3}+126 x^{2}+216 x & =18 x\left(x^{2}+7 x+12\right) \\
& =18 x(x+3)(x+4) \\
15 x^{4}-30 x^{3}-15 x^{2} & =15 x^{2}\left(x^{2}-2 x-15\right) \\
& =15 x^{2}(x+3)(x-5) \\
42 x^{5}-42 x^{4}-840 x^{3} & =42 x^{3}\left(x^{2}-x-20\right) \\
& =42 x^{3}(x+4)(x-5)
\end{aligned}
$$

$$
\begin{array}{rlllllllll}
18 x^{3}+126 x^{2}+216 x & =2 & 3 & 3 & & x & & (x+3) & (x+4) \\
15 x^{4}-30 x^{3}-15 x^{2} & = & 3 & 5 & & x & x & (x+3) & (x-5) \\
42 x^{5}-42 x^{4}-840 x^{3} & =2 & 3 & 5 & 7 & x & x & x & (x+4)(x-5) \\
\hline \text { LCM } & =2 & 3 & 3 & 5 & 7 & x & x & x(x+3) & (x+4)(x-5)
\end{array}
$$

What do you multiply $18 x^{3}+126 x^{2}+\mathbf{2 1 6 x}$ by to get the LCM?
Answer: $\mathbf{3 5} \boldsymbol{x}^{\mathbf{2}}(\boldsymbol{x}-5)$
What do you multiply $15 x^{4}-30 x^{3}-15 x^{2}$ by to get the LCM?
Answer: $\mathbf{4 2 x}(\boldsymbol{x}+4)$
What do you multiply $42 x^{5}-42 x^{4}-840 x^{3}$ by to get the LCM?
Answer: 3( $\boldsymbol{x}+\mathbf{3}$ )

## Chapter 48

# Addition/Subtraction of Rational Expressions (Unlike Denominators) 

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### 48.1 Youtube

https://www.youtube.com/playlist?list=PLC2291951FF3FA3F5\&feature=view_all

### 48.2 Basics

We saw that adding and subtracting fractions required the same denominator. If we use the Least Common Denominator (LCD), then we work with the smallest possible numbers (expressions).



Think of one "orange" as $\frac{\mathbf{1}}{\mathbf{2}}$ of the rectangle, one "apple" as $\frac{\mathbf{1}}{\mathbf{3}}$ of the rectangle, and one "banana" as $\frac{\mathbf{1}}{\mathbf{6}}$ of the rectangle,

The crucial meaning of the common denominator is that we transformed apples and oranges to bananas. We cannot add apples to oranges, but we add bananas to bananas. It is absolutely imperative that you understand this concept of common denominator.
$\frac{1}{3}=\frac{1}{3} \cdot 1=\frac{1}{3} \cdot \frac{2}{2}=\frac{2}{6}$ (Two bananas)
$\frac{1}{2}=\frac{1}{2} \cdot 1=\frac{1}{2} \cdot \frac{3}{3}=\frac{3}{6}$ (Three bananas)
$\frac{1}{3}+\frac{1}{2}=\frac{2}{6}+\frac{3}{6}=\frac{3+2}{6}=\frac{5}{6}$ (Five bananas)

### 48.3 Examples

Example 1:
Add $\frac{\mathbf{2}}{\mathbf{5}}+\frac{\mathbf{3}}{\mathbf{4}}$

## Solution:

$\frac{2}{5}=\frac{2}{5} \cdot \frac{4}{4}=\frac{8}{20}$
$\frac{3}{4}=\frac{3}{4} \cdot \frac{5}{5}=\frac{15}{20}$
$\frac{2}{5}+\frac{3}{4}=\frac{8}{20}+\frac{15}{20}=\frac{8+15}{20}=\frac{23}{20}$. The result can also be written $1 \frac{3}{20}$

Example 2:
Add $\frac{\boldsymbol{a}}{\boldsymbol{b}}+\frac{\boldsymbol{c}}{\boldsymbol{d}}$
Solution:
$\frac{a}{b}=\frac{a}{b} \cdot \frac{d}{d}=\frac{a d}{b d}$
$\frac{c}{d}=\frac{c}{d} \cdot \frac{b}{b}=\frac{b c}{b d}$
$\frac{a}{b}+\frac{c}{d}=\frac{a d}{b d}+\frac{b c}{b d}=\frac{a d+b c}{b d}$.
Example 3:
Add $\frac{2}{15}+\frac{3}{20}$
Solution:
$\frac{2}{15}=\frac{2}{15} \cdot \frac{20}{20}=\frac{40}{300}$
$\frac{3}{20}=\frac{3}{20} \cdot \frac{15}{15}=\frac{45}{300}$
$\frac{2}{15}+\frac{3}{20}=\frac{40}{300}+\frac{45}{300}=\frac{85}{300}=\frac{17 \cdot 5}{60 \cdot 5}=\frac{17}{60} \cdot 1=\frac{17}{60}$.
The object of the development above is to show the importance of the common denominator. The development below will use the LCD which is $\mathbf{6 0}$.

| 15 | $=$ |  | 3 | 5 |  |  |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 20 | $=$ | 2 |  | 5 |  |  |
| LCD | $=$ | 2 | 3 | 5 | $=$ | 60 |

What do you multiply 15 by to get 60? Answer: $2 \cdot 2=4$ (what factors are missing in the array above for the line containing $\mathbf{1 5}$ ?)

What to multiply 20 by to get 60? Answer: $\mathbf{3}$ (what factor is missing in the array above for the line containing 20?)
$\frac{2}{15}=\frac{2}{15} \cdot \frac{4}{4}=\frac{8}{60}$
$\frac{3}{20}=\frac{3}{20} \cdot \frac{3}{3}=\frac{9}{60}$
$\frac{2}{15}+\frac{3}{20}=\frac{8}{60}+\frac{9}{60}=\frac{17}{60}$.
The result is identical in both developments.

Example 4:
Add $\frac{4}{x-2}+\frac{2 \boldsymbol{x}}{2-\boldsymbol{x}}$
Solution: (not recommended)

$$
\begin{aligned}
\frac{4}{x-2}+\frac{2 x}{2-x} & =\frac{4}{x-2} \cdot \frac{2-x}{2-x}+\frac{2 x}{2-x} \cdot \frac{x-2}{x-2} \\
& =\frac{8-4 x}{(x-2)(2-x)}+\frac{2 x^{2}-4 x}{(2-x)(x-2)} \\
& =\frac{8-4 x+2 x^{2}-4 x}{(x-2)(2-x)} \\
& =\frac{2 x^{2}-8 x+8}{(x-2)(2-x)} \\
& =\frac{2\left(x^{2}-4 x+4\right)}{(x-2)(2-x)} \\
& =\frac{2(x-2)^{2}}{(x-2)(2-x)} \\
& =\frac{2(x-2)}{(2-x)} \\
& =\frac{2(x-2)}{-(x-2)} \\
& =\frac{2}{-1}=-2
\end{aligned}
$$

The object of the development above is to show the importance of the common denominator. The development below will use the LCD which is $\boldsymbol{x} \mathbf{- 2}$.

$$
\begin{aligned}
\frac{4}{x-2}+\frac{2 x}{2-x} & =\frac{4}{x-2}+\frac{2 x}{-(x-2)} \\
& =\frac{4}{x-2}-\frac{2 x}{x-2} \\
& =\frac{4-2 x}{x-2} \\
& =\frac{2(2-x)}{x-2} \\
& =\frac{2(-1)(x-2)}{x-2} \\
& =\frac{-2}{1}=-2
\end{aligned}
$$

The result is identical in both developments.

### 48.3. EXAMPLES

Example 5:
Simplify $\frac{2 x}{x^{2}+3 x-10}+\frac{4 x}{2 x^{2}-7 x+6}-\frac{8 x}{2 x^{2}+7 x-15}$

## Solution:

$$
\begin{aligned}
& D_{1}=x^{2}+3 x-10=(x+5)(x-2) \\
& D_{2}=2 x^{2}-7 x+6 \\
& \\
& \qquad \begin{array}{rl}
P=12 \\
-1 & S=-7 \\
-2 \\
-3 & -12 \\
2 x^{2}-7 x+6 & =2 x^{2}-3 x-4 x+6 \\
& =x(2 x-3)-2(2 x-3) \\
& =(2 x-3)(x-2)
\end{array}
\end{aligned}
$$

$$
D_{3}=2 x^{2}+7 x-15
$$

$$
P=-30
$$

$$
S=7
$$

$$
\begin{array}{l|ll}
-1 & 30 & \\
-2 & 15 & \\
-3 & 10 & \text { sum }=7
\end{array}
$$

$$
2 x^{2}+7 x-15==2 x^{2}-3 x+10 x-15
$$

$$
=x(2 x-3)+5(2 x-3)
$$

$$
=(2 x-3)(x+5)
$$

$$
x^{2}+3 x-10=(x+5) \quad(x-2)
$$

$$
2 x^{2}-7 x+6=\quad(x-2) \quad(2 x-3)
$$

$$
2 x^{2}+7 x-15=(x+5)
$$

$$
\mathrm{LCD}=(x+5) \quad(x-2) \quad(2 x-3)
$$

$$
\frac{2 x}{x^{2}+3 x-10}+\frac{4 x}{2 x^{2}-7 x+6}-\frac{8 x}{2 x^{2}+7 x-15}
$$

$$
=\frac{2 x}{(x+5)(x-2)}+\frac{4 x}{(2 x-3)(x-2)}-\frac{8 x}{(2 x-3)(x+5)}
$$

$$
=\frac{2 x(2 x-3)}{(x+5)(x-2)(2 x-3)}+\frac{4 x(x+5)}{(2 x-3)(x-2)(x+5)}-\frac{8 x(x-2)}{(2 x-3)(x+5)(x-2)}
$$

$$
=\frac{4 x^{2}-6 x}{(x+5)(x-2)(2 x-3)}+\frac{4 x^{2}+20 x}{(2 x-3)(x-2)(x+5)}-\frac{8 x^{2}-16 x}{(2 x-3)(x+5)(x-2)}
$$

$$
=\frac{4 x^{2}-6 x+4 x^{2}+20 x-8 x^{2}+16 x}{(x+5)(x-2)(2 x-3)}=\frac{30 x}{(x+5)(x-2)(2 x-3)}
$$

Example 6:
Simplify $\frac{x}{x^{2}-25}+\frac{x}{x-5}-\frac{x}{x+5}$

## Solution:

$$
\begin{aligned}
& \frac{x}{x^{2}-25}+\frac{x}{x-5}-\frac{x}{x+5} \\
= & \frac{x}{(x+5)(x-5)}+\frac{x}{x-5}-\frac{x}{x+5} \\
= & \frac{x}{(x+5)(x-5)}+\frac{x(x+5)}{(x-5)(x+5)}-\frac{x(x-5)}{(x+5)(x-5)} \\
= & \frac{x}{(x+5)(x-5)}+\frac{x^{2}+5 x}{(x-5)(x+5)}-\frac{x^{2}-5 x}{(x+5)(x-5)} \\
= & \frac{x+x^{2}+5 x-\left(x^{2}-5 x\right)}{(x+5)(x-5)} \\
= & \frac{x+x^{2}+5 x-x^{2}+5 x}{(x+5)(x-5)} \\
= & \frac{11 x}{(x+5)(x-5)}
\end{aligned}
$$

### 48.4 Exercise 48

1. $\operatorname{Add} \frac{5}{7}+\frac{7}{5}$
2. $\operatorname{Add} \frac{\boldsymbol{p}}{\boldsymbol{q}}+\frac{\boldsymbol{r}}{\boldsymbol{s}}$
3. Add $\frac{5}{12}+\frac{7}{30}$
4. Add $\frac{5 x}{x-5}+\frac{\boldsymbol{x}^{2}}{5-\boldsymbol{x}}$
5. Simplify $\frac{3}{x^{2}-3 x-28}+\frac{2}{2 x^{2}+13 x+20}-\frac{8}{2 x^{2}-9 x-35}$
6. Simplify $\frac{7}{x^{2}-49}+\frac{2}{x-7}-\frac{5}{x+7}$

## STOP!

1. Add $\frac{\mathbf{5}}{\mathbf{7}}+\frac{\mathbf{7}}{\mathbf{5}}$

## Solution:

$\frac{5}{7}=\frac{5}{7} \cdot \frac{5}{5}=\frac{25}{35}$
$\frac{7}{5}=\frac{7}{5} \cdot \frac{7}{7}=\frac{49}{35}$
$\frac{5}{7}+\frac{7}{5}=\frac{25}{35}+\frac{49}{35}=\frac{25+49}{35}=\frac{74}{35}$.
The result can also be written $2 \frac{4}{35}$
2. $\operatorname{Add} \frac{\boldsymbol{p}}{\boldsymbol{q}}+\frac{\boldsymbol{r}}{\boldsymbol{s}}$

## Solution:

$\frac{p}{q}=\frac{p}{q} \cdot \frac{s}{s}=\frac{p s}{q s}$
$\frac{r}{s}=\frac{r}{s} \cdot \frac{q}{q}=\frac{r q}{q s}$
$\frac{p}{q}+\frac{r}{s}=\frac{p s}{q s}+\frac{r q}{q s}=\frac{p s+r q}{q s}$.
3. Add $\frac{5}{12}+\frac{7}{30}$

Solution:
$\frac{5}{12}+\frac{7}{30}$
$\frac{5}{12}=\frac{5}{12} \cdot \frac{30}{30}=\frac{150}{360}$
$\frac{7}{30}=\frac{7}{30} \cdot \frac{12}{12}=\frac{84}{360}$
$\frac{5}{12}+\frac{7}{30}=\frac{150}{360}+\frac{84}{360}=\frac{234}{360}=\frac{6 \cdot 39}{6 \cdot 60}=1 \cdot \frac{39}{60}=\frac{39}{60}=\frac{13}{20}$.
The object of the development above is to show the importance of the common denominator. The development below will use the LCD which is $\mathbf{6 0}$.

| 12 | $=$ | 2 | 3 |  |  |  |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 30 | $=$ | 2 | 3 | 5 |  |  |
| LCD | $=$ | 2 | 3 | 5 | $=$ | 60 |

What do you multiply 12 by to get 60? Answer: 5 (what factor(s) is(are) missing in the array above for the line containing 12?)
What do you multiply $\mathbf{3 0}$ by to get $\mathbf{6 0}$ ? Answer: 2 (missing factor in the above line of $\mathbf{3 0}$ ?)

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$\frac{5}{12}=\frac{5}{12} \cdot \frac{5}{5}=\frac{25}{60}$
$\frac{7}{30}=\frac{7}{30} \cdot \frac{2}{2}=\frac{14}{60}$
$\frac{5}{12}+\frac{7}{30}=\frac{\mathbf{2 5}}{60}+\frac{14}{60}=\frac{\mathbf{3 9}}{60}=\frac{13}{20}$. The result is identical in both developments.
4. Add $\frac{5 x}{x-5}+\frac{x^{2}}{5-x}$

Solution: (not recommended)

$$
\begin{aligned}
\frac{5 x}{x-5}+\frac{x^{2}}{5-x} & =\frac{5 x}{x-5} \cdot \frac{5-x}{5-x}+\frac{x^{2}}{5-x} \cdot \frac{x-5}{x-5} \\
& =\frac{25 x-5 x^{2}}{(x-5)(5-x)}+\frac{x^{3}-5 x^{2}}{(5-x)(x-5)} \\
& =\frac{25 x-5 x^{2}+x^{3}-5 x^{2}}{(x-5)(5-x)} \\
& =\frac{x^{3}-10 x^{2}+25 x}{(x-5)(5-x)} \\
& =\frac{x\left(x^{2}-10 x+25\right)}{(x-5)(5-x)} \\
& =\frac{x(x-5)^{2}}{(x-5)(5-x)} \\
& =\frac{x(x-5)}{5-x} \\
& =\frac{x(x-5)}{-(x-5)}=\frac{x}{-1}=-x
\end{aligned}
$$

The object of the development above is to show the importance of the common denominator. The development below will use the LCD which is $\boldsymbol{x}-\mathbf{5}$.

$$
\begin{aligned}
\frac{5 x}{x-5}+\frac{x^{2}}{5-x} & =\frac{5 x}{-(5-x)}+\frac{x^{2}}{5-x} \\
& =\frac{-5 x}{5-x}+\frac{x^{2}}{5-x} \\
& =\frac{x^{2}-5 x}{5-x} \\
& =\frac{x(x-5)}{5-x} \\
& =\frac{x(x-5)}{-(x-5)} \\
& =\frac{x}{-1}=-x
\end{aligned}
$$

The result is identical in both developments.
5. Simplify $\frac{3}{x^{2}-3 x-28}+\frac{2}{2 x^{2}+13 x+20}-\frac{8}{2 x^{2}-9 x-35}$

## Solution:

$$
D_{1}=x^{2}-3 x-28=(x-7)(x+4)
$$

$$
D_{3}=2 x^{2}-9 x-35
$$

$$
P=-70
$$

$$
S=-9
$$

1

| 1 |  |
| :--- | :--- |
| 2 | -35 |
| 5 |  |$\quad \begin{aligned} & -14\end{aligned}$ sum $=5-14=-9$

$$
\begin{aligned}
2 x^{2}-9 x-35 & =2 x^{2}+5 x-14 x-35 \\
& =x(2 x+5)-7(2 x+5) \\
& =(2 x+5)(x-7)
\end{aligned}
$$

$$
\begin{array}{llll}
x^{2}-3 x-28 & = & (x-7) & (x+4) \\
2 x^{2}+13 x+20 & = & (x+4) & (2 x+5) \\
2 x^{2}-9 x-35 & =(x-7) & & (2 x+5) \\
\hline \text { LCD } & =(x-7) & (x+4) & (2 x+5)
\end{array}
$$

$$
\begin{aligned}
& D_{2}=2 x^{2}+13 x+20 \\
& P=40 \\
& S=13 \\
& 1 \text { | } 40 \\
& 2 \text { | } 20 \\
& \begin{array}{l|ll}
4 & 10 \\
8 & 5 & \text { sum }=8+5=13
\end{array} \\
& 2 x^{2}+13 x+20=2 x^{2}+8 x+5 x+20 \\
& =2 x(x+4)+5(x+4) \\
& =(x+4)(2 x+5)
\end{aligned}
$$

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$$
\begin{aligned}
& \frac{3}{x^{2}-3 x-28}+\frac{2}{2 x^{2}+13 x+20}-\frac{8}{2 x^{2}-9 x-35} \\
= & \frac{3}{(x-7)(x+4)}+\frac{2}{(x+4)(2 x+5)}-\frac{8}{(2 x+5)(x-7)} \\
= & \frac{3(2 x+5)}{(x-7)(x+4)(2 x+5)}+\frac{2(x-7)}{(x+4)(2 x+5)(x-7)}-\frac{8(x+4)}{(2 x+5)(x-7)(x+4)} \\
= & \frac{6 x+15}{(x-7)(x+4)(2 x+5)}+\frac{2 x-14}{(x+4)(2 x+5)(x-7)}-\frac{8 x+32}{(2 x+5)(x-7)(x+4)} \\
= & \frac{6 x+15+2 x-14-(8 x+32)}{(x-7)(x+4)(2 x+5)} \\
= & \frac{6 x+15+2 x-14-8 x-32}{(x-7)(x+4)(2 x+5)} \\
= & \frac{-31}{(x-7)(x+4)(2 x+5)}
\end{aligned}
$$

6. Simplify $\frac{7}{x^{2}-49}+\frac{2}{x-7}-\frac{5}{x+7}$

## Solution:

$$
\begin{aligned}
& \frac{7}{x^{2}-49}+\frac{2}{x-7}-\frac{5}{x+7} \\
= & \frac{7}{(x-7)(x+7)}+\frac{2}{x-7}-\frac{5}{x+7} \\
= & \frac{7}{(x-7)(x+7)}+\frac{2(x+7)}{(x-7)(x+7)}-\frac{5(x-7)}{(x+7)(x-7)} \\
= & \frac{7}{(x-7)(x+7)}+\frac{2 x+14}{(x-7)(x+7)}-\frac{5 x-35}{(x+7)(x-7)} \\
= & \frac{7+2 x+14-(5 x-35)}{(x-7)(x+7)} \\
= & \frac{7+2 x+14-5 x+35}{(x-7)(x+7)} \\
= & \frac{56-3 x}{(x-7)(x+7)}
\end{aligned}
$$

## Chapter 49

# Complex Rational Expressions) 

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### 49.1 Youtube

https://www.youtube.com/playlist?list=PL53D60310218C37C2\&feature=view_all

### 49.2 Basics

Example 1:
$\frac{\frac{1}{2}+\frac{3}{4}}{\frac{5}{6}+\frac{7}{8}}$

## Solution:

Method 1: Combine numerator/denominator fractions individually.
$\frac{\frac{1}{2}+\frac{3}{4}}{\frac{5}{6}+\frac{7}{8}}=\frac{\frac{2}{4}+\frac{3}{4}}{\frac{5 \cdot 4}{6 \cdot 4}+\frac{7 \cdot 3}{8 \cdot 3}}=\frac{\frac{2+3}{4}}{\frac{20+21}{24}}=\frac{\frac{5}{4}}{\frac{41}{24}}=\frac{5}{4} \cdot \frac{24}{41}=\frac{5}{1} \cdot \frac{6}{41}=\frac{30}{41}$
Method 2: Multiply numerator and denominator by the LCD of all the denominators.

| 2 | $=$ | 2 |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 4 | $=$ | 2 | 2 |  |
| 6 | $=$ | 2 |  |  |
| 8 | $=$ | 2 | 2 | 2 |
| LCD | $=$ | 2 | 2 |  |

$\frac{\frac{1}{2}+\frac{3}{4}}{\frac{5}{6}+\frac{7}{8}}=\frac{\left(\frac{1}{2}+\frac{3}{4}\right)\left(\frac{24}{1}\right)}{\left(\frac{5}{6}+\frac{7}{8}\right)\left(\frac{24}{1}\right)}=\frac{\frac{1 \cdot 24}{2}+\frac{3 \cdot 24}{4}}{\frac{5 \cdot 24}{6}+\frac{7 \cdot 24}{8}}=\frac{\frac{1 \cdot 12}{1}+\frac{3 \cdot 6}{1}}{\frac{5 \cdot 4}{1}+\frac{7 \cdot 3}{1}}=\frac{12+18}{20+21}=\frac{30}{41}$
This second method is easier and quicker.
Example 2:
Simplify $\frac{3-\frac{x}{5}}{9-\frac{x^{2}}{25}}$

## Solution:

$$
\begin{aligned}
\frac{3-\frac{x}{5}}{9-\frac{x^{2}}{25}} & =\frac{\left(3-\frac{x}{5}\right)\left(\frac{25}{1}\right)}{\left(9-\frac{x^{2}}{25}\right)\left(\frac{25}{1}\right)} \\
& =\frac{3 \cdot 25-\frac{x \cdot 25}{5}}{9 \cdot 25-\frac{x^{2} \cdot 25}{25}} \\
& =\frac{75-\frac{x \cdot 5}{1}}{225-\frac{x^{2}}{1}}=\frac{5(15-x)}{(15-x)(15+x)}=\frac{5}{15+x}
\end{aligned}
$$

Note: The denominator is a difference of squares.
Alternate method for this special example:

$$
\begin{aligned}
\frac{3-\frac{x}{5}}{9-\frac{x^{2}}{25}} & =\frac{3-\frac{x}{5}}{\left(3+\frac{x}{5}\right)\left(3-\frac{x}{5}\right)} \\
& =\frac{1}{3+\frac{x}{5}}=\frac{1 \cdot 5}{3 \cdot 5+\frac{x \cdot 5}{5}}=\frac{5}{15+x}
\end{aligned}
$$

Example 3:
Simplify $\frac{\frac{5}{x-3}-\frac{7}{x-2}}{\frac{6}{x-2}-\frac{4}{x-3}}$

## Solution:

Multiply numerator and denominator by the LCD $(x-2)(x-3)$ and distribute.

$$
\begin{aligned}
\frac{\frac{5}{x-3}-\frac{7}{x-2}}{\frac{6}{x-2}-\frac{4}{x-3}} & =\frac{\frac{5(x-3)(x-2)}{x-3}-\frac{7(x-3)(x-2)}{x-2}}{\frac{6(x-3)(x-2)}{x-2}-\frac{4(x-3)(x-2)}{x-3}} \\
& =\frac{5(x-2)-7(x-3)}{6(x-3)-4(x-2)} \\
& =\frac{5 x-10-7 x+21}{6 x-18-4 x+8}=\frac{-2 x+11}{2 x-10}
\end{aligned}
$$

### 49.3 Exercise 49

1. Simplify $\frac{\frac{5}{6}+\frac{7}{9}}{\frac{11}{12}+\frac{2}{14}}$
2. Simplify $\frac{5+\frac{\boldsymbol{y}}{7}}{25-\frac{y^{2}}{49}}$
3. Simplify $\frac{\frac{6}{x-5}-\frac{9}{x+4}}{\frac{2}{x+4}+\frac{3}{x-5}}$

## STOP!

1. $\frac{\frac{5}{6}+\frac{7}{9}}{\frac{11}{12}+\frac{2}{14}}$

## Solution:

Method 1: Combine fractions in the numerator and denominator individually.

$$
\begin{aligned}
\frac{\frac{5}{6}+\frac{7}{9}}{\frac{11}{12}+\frac{2}{14}} & =\frac{\frac{5 \cdot 3}{6 \cdot 3}+\frac{7 \cdot 2}{9 \cdot 2}}{\frac{11 \cdot 7}{12 \cdot 7}+\frac{2 \cdot 6}{14 \cdot 6}} \\
& =\frac{\frac{15+14}{18}}{\frac{77+12}{84}} \\
& =\frac{\frac{29}{18}}{\frac{89}{84}}=\frac{29}{18} \cdot \frac{84}{89}=\frac{29}{6 \cdot 3} \cdot \frac{6 \cdot 14}{89}=\frac{406}{267}
\end{aligned}
$$

Method 2: Multiply numerator and denominator by the LCD of all the denominators.

| 6 | $=$ | 2 | 3 |  |  |
| ---: | :--- | :--- | :--- | :--- | :--- |
| 9 |  |  | 3 | 3 |  |
| 12 | $=$ | 2 | 3 |  |  |
| 14 | $=$ | 2 |  | 7 |  |
| LCD | $=$ | 2 | 2 | 3 | 3 |
|  | $7=252$ |  |  |  |  |

$$
\begin{aligned}
\frac{\frac{5}{6}+\frac{7}{9}}{\frac{11}{12}+\frac{2}{14}} & =\frac{\left(\frac{5}{6}+\frac{7}{9}\right)\left(\frac{252}{1}\right)}{\left(\frac{11}{12}+\frac{2}{14}\right)\left(\frac{252}{1}\right)} \\
& =\frac{\frac{5 \cdot 252}{6}+\frac{7 \cdot 252}{9}}{\frac{11 \cdot 252}{12}+\frac{2 \cdot 252}{14}} \\
& =\frac{\frac{5 \cdot 42}{1}+\frac{7 \cdot 28}{1}}{\frac{11 \cdot 21}{1}+\frac{2 \cdot 18}{1}}=\frac{210+196}{231+36}=\frac{406}{267}
\end{aligned}
$$

$5+\frac{\boldsymbol{y}}{\boldsymbol{7}}$
2. Simplify $\frac{7}{25-\frac{y^{2}}{49}}$

$$
\begin{aligned}
& \begin{array}{l}
\text { Solution: } \\
25+\frac{y}{7} \\
25-\frac{y^{2}}{49}
\end{array}=\frac{\left(5+\frac{y}{7}\right)\left(\frac{49}{1}\right)}{\left(25-\frac{y^{2}}{49}\right)\left(\frac{49}{1}\right)} \\
& =\frac{5 \cdot 49-\frac{y \cdot 49}{7}}{25 \cdot 49-\frac{y^{2} \cdot 49}{49}} \\
& =\frac{245-\frac{y \cdot 7}{1}}{1225-\frac{y^{2}}{1}}=\frac{7(35-y)}{(35-y)(35+y)}=\frac{7}{35+y}
\end{aligned}
$$

Note: The denominator is a difference of squares.

$$
\frac{5+\frac{y}{7}}{25-\frac{y^{2}}{49}}=\frac{5+\frac{y}{7}}{\left(5+\frac{y}{7}\right)\left(5-\frac{y}{7}\right)}=\frac{1}{5-\frac{y}{7}}=\frac{1 \cdot 7}{5 \cdot 7+\frac{y \cdot 7}{7}}=\frac{7}{35+y}
$$

3. Simplify $\frac{\frac{6}{x-5}-\frac{9}{x+4}}{\frac{2}{x+4}+\frac{3}{x-5}}$

Solution:

$$
\begin{aligned}
\frac{\frac{6}{x-5}-\frac{9}{x+4}}{\frac{2}{x+4}+\frac{3}{x-5}} & =\frac{\frac{6(x-5)(x+4)}{x-5}-\frac{9(x-5)(x+4)}{x+4}}{\frac{2(x-5)(x+4)}{x+4}+\frac{3(x-5)(x+4)}{x-5}} \\
& =\frac{6(x+4)-9(x-5)}{2(x-5)+3(x+4)} \\
& =\frac{6 x+24-9 x+45}{2 x-10+3 x+12} \\
& =\frac{-3 x+69}{5 x+2}=-\frac{3(x-23)}{5 x+2}
\end{aligned}
$$

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## Chapter 50

## Solving Equations with Rational Expressions)

### 50.1 Youtube

https://www.youtube.com/playlist?list=PL0F0B6BF9036A6B1D\&feature=view_all

### 50.2 Basics

Think of an equation as a scale in equilibrium.
You can add the same number (expression) to both sides.
You can subtract the same number (expression) from both sides.
You can multiply both sides by the same number (and then distribute if there are more terms).
You can divide both sides by the same number (and then distribute if there are more terms).
The objective is to end up with a single term involving a variable on one side of the equal sign and one onstant on the other side.

The next step is to get one tomes thevariable and the linear equation is solved.

### 50.3 Examples

Example 1:
Solve $\frac{5}{x+3}+\frac{6}{x-3}=\frac{80}{x^{2}-9}$

## Solution:

$$
\begin{aligned}
\frac{5}{x+3}+\frac{6}{x-3} & =\frac{80}{x^{2}-9} \\
\frac{5(x+3)(x-3)}{x+3}+\frac{6(x+3)(x-3)}{x-3} & =\frac{80(x+3)(x-3)}{x^{2}-9}
\end{aligned}
$$

Multiply both sides by the $\mathrm{LCD}=(\boldsymbol{x}+\mathbf{3})(\boldsymbol{x}-\mathbf{3})$

$$
\begin{aligned}
& 5(x-3)+6(x+3)=80 \\
& \text { reduce } \\
& 5 x-15+6 x+18=80 \\
& \text { distribute } \\
& 11 x+3=80 \quad \text { combine like terms } \\
& 11 x=77 \text { divide by } 11 \\
& x=\frac{77}{11}=7
\end{aligned}
$$

We must verify that no denominator is 0 to avoid division by 0 .
$x+3=7+3 \neq 0$
$x-3=7-3 \neq 0$
$x^{2}-9=7^{2}-9 \neq 0$
Then $\boldsymbol{x}=\mathbf{7}$ is a solution (assuming the steps are correct).
Example 2:
Solve $\frac{4 x}{x-4}=3+\frac{16}{x-4}$

## Solution:

$$
\begin{aligned}
\frac{4 x}{x-4} & =3+\frac{16}{x-4} & & \\
\frac{4 x(x-4)}{x-4} & =3(x-4)+\frac{16(x-4)}{x-4} & & \text { distribute }(x-4) \\
4 x & =3 x-12+16 & & \text { reduce } \\
x & =4 & & \text { subtract } 3 x \text { from both sides. }
\end{aligned}
$$

We must verify that no denominator is 0 to avoid division by 0 .
$x-4=4-4=0 \quad$ Thus $x=4$ is not a solution.
This example has no solution.
Example 3:
Solve $\frac{3 x}{x-4}=3+\frac{16}{x-4}$

## Solution:

$$
\begin{aligned}
\frac{3 x}{x-4} & =3+\frac{16}{x-4} & & \\
\frac{3 x(x-4)}{x-4} & =3(x-4)+\frac{16(x-4)}{x-4} & & \text { distribute }(x-4) \\
3 x & =3 x-12+16 & & \text { reduce } \\
0 x & =-12+16 & & \text { subtract } 3 x \text { from both sides. } \\
0 & =4 & & \text { False statement can't be corrected for any } x .
\end{aligned}
$$

This example has no solution.
Example 4:
Solve $\frac{3 x}{x-4}=3+\frac{12}{x-4}$

## Solution:

$$
\begin{aligned}
\frac{3 x}{x-4} & =3+\frac{12}{x-4} & & \\
\frac{3 x(x-4)}{x-4} & =3(x-4)+\frac{12(x-4)}{x-4} & & \text { distribute }(x-4) \\
3 x & =3 x-12+12 & & \text { reduce } \\
0 x & =0 & & \text { subtract } 3 x \text { from both sides. } \\
0 & =0 & & \text { True statement cannot be falsified for any } x .
\end{aligned}
$$

This example has infinitely many solutions. Any real number other than $\boldsymbol{x}=4$ is a solution.

## Example 5:

A car and a truck leave a park at the same time. The speed of the car is $\mathbf{2 0}$ miles per hour more than that of the truck. The truck's destination is $\mathbf{2 1 0}$ miles. The car's destination is $\mathbf{1 8 6}$ miles. The car reaches its destination in $\mathbf{2}$ hours less than the truck. What is the speed of the truck?

## Solution:

Let $\boldsymbol{x}$ be the speed of the truck.
Then $\boldsymbol{x}+\mathbf{2 0}$ is the speed of the car.

|  | Rate | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Truck | $\boldsymbol{x}$ | $\boldsymbol{T}_{\boldsymbol{T}}$ | $\mathbf{2 1 0}$ |
| Car | $\boldsymbol{x}+\mathbf{2 0}$ | $\boldsymbol{T}_{\boldsymbol{C}}$ | $\mathbf{1 8 6}$ |

Since $\boldsymbol{R T}=\boldsymbol{D}, \boldsymbol{T}=\frac{\boldsymbol{D}}{\boldsymbol{R}}$
The Time the truck traveled was $\boldsymbol{T}_{\boldsymbol{T}}=\frac{\mathbf{2 1 0}}{\boldsymbol{x}}$ and the Time the car traveled was $\boldsymbol{T}_{\boldsymbol{C}}=\frac{\mathbf{1 8 6}}{\boldsymbol{x}+\mathbf{2 0}}$
The car reaches its destination in 2 hours less than the truck, so

$$
\begin{aligned}
& T_{T}=T_{C}+2 \\
& \frac{210}{x}=\frac{186}{x+20}+2 \\
& \frac{210 x(x+20)}{x}=\frac{186 x(x+20)}{x+20}+2 x(x+20) \\
& 210(x+20)=186 x+2 x(x+20) \\
& 210 x+4200=186 x+2 x^{2}+40 x \\
& 2 x^{2}+16 x-4200=0 \\
& x^{2}+8 x-2100=0 \\
& P=-2100 \\
& S=8
\end{aligned}
$$

$$
\begin{aligned}
x^{2}+8 x-2100 & =0 \\
x^{2}-42 x+50 x-2100 & =0 \\
x(x-42)+50(x-42) & =0 \\
(x-42)(x+50) & =0
\end{aligned}
$$

We reject $\boldsymbol{x}=\mathbf{- 5 0}$ because speed cannot be negative.
The speed of the truck is $\boldsymbol{x}=\mathbf{4 2}$ miles per hour.

## Example 6:

A small pipe can fill a tank in 14 hours more than a large pipe. Together they fill the tank in 24 hours. How many hours does it take the big pipe to fill the tank by itself?

## Solution:

Let $\boldsymbol{x}$ be the number of hours it takes the large pipe to fill the tank.
Then $\boldsymbol{x}+\mathbf{1 4}$ is the number of hours it takes the small pipe to fill the tank.

|  | Total time to fill the tank | Portion of tank filled per hour |
| :---: | :---: | :---: |
| Large pipe | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| Small pipe | $\boldsymbol{x}+\mathbf{1 4}$ | $\frac{\mathbf{1}}{\boldsymbol{x + 1 4}}$ |
| Both pipes | $\mathbf{2 4}$ | $\frac{\mathbf{1}}{\mathbf{2 4}}$ |

The portion of the tank filled by the large pipe added to the portion of the tank filled by the small pipe in 1 hour equals the portion of the tank filled by both pipes in 1 hour. Thus

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{x+14} & =\frac{1}{24} \\
\frac{24 x(x+14)}{x}+\frac{24 x(x+14)}{x+14} & =\frac{24 x(x+14)}{24} \\
24(x+14)+24 x & =x(x+14) \\
24 x+336+24 x & =x^{2}+14 x \\
0 & =x^{2}-34 x-336
\end{aligned}
$$

```
P=-336
    S=-34
        | -336
        | -168
        | -112
        -84
        | -56
        -56
        -42 sum = 8-42=-34
```

$$
\begin{aligned}
x^{2}-34 x-336 & =0 \\
x^{2}+8 x-42 x-336 & =0 \\
x(x+8)-42(x+8) & =0 \\
(x+8)(x-42) & =0
\end{aligned}
$$

Reject $\boldsymbol{x}=\mathbf{- 8}$ because a pipe cannot be on for a negative time.
Thus it takes 42 hours for the large pipe to fill the tank.

### 50.4 Exercise 50

1. Solve $\frac{7}{x+5}-\frac{2}{x-5}=\frac{15}{x^{2}-25}$
2. Solve $\frac{7 x}{x-2}=3+\frac{14}{x-2}$
3. Solve $\frac{7 x}{x-2}=7+\frac{16}{x-2}$
4. Solve $\frac{7 x}{x-2}=7+\frac{14}{x-2}$
5. A car and a truck leave a park at the same time. The speed of the car is 40 miles per hour more than that of the truck. The truck's destination is $\mathbf{1 1 5}$ miles. The car's destination is $\mathbf{1 2 6}$ miles. The car reaches its destination in $\mathbf{3}$ hours less than the truck. What is the speed of the truck?
6. A small pipe can fill a tank in $\mathbf{1 7}$ hours more than a large pipe. Together they fill the tank in $\mathbf{7 2}$ hours. How many hours does it take the big pipe to fill the tank by itself?

## STOP!

1. Solve $\frac{7}{x+5}-\frac{2}{x-5}=\frac{15}{x^{2}-25}$

## Solution:

$$
\begin{aligned}
\frac{7}{x+5}-\frac{2}{x-5} & =\frac{15}{x^{2}-25} \\
\frac{7(x+5)(x-5)}{x+5}-\frac{2(x+5)(x-5)}{x-5} & =\frac{15(x+5)(x-5)}{x^{2}-25} \\
7(x-5)-2(x+5) & =15 \\
7 x-35-2 x-10 & =15 \\
5 x-45 & =15 \\
5 x & =60 \\
x & =\frac{60}{2}=12
\end{aligned}
$$

We must verify that no denominator is 0 to avoid division by 0 .

$$
\begin{aligned}
& x+5=12+5 \neq 0 \\
& x-5=12-5 \neq 0 \\
& x^{2}-25=12^{2}-25 \neq 0
\end{aligned}
$$

2. Solve $\frac{7 x}{x-2}=3+\frac{14}{x-2}$

Then $\boldsymbol{x}=\mathbf{1 2}$ is a solution (assuming the arithmetic is correct).

## Solution:

$$
\begin{aligned}
\frac{7 x}{x-2} & =3+\frac{14}{x-2} \\
\frac{7 x(x-2)}{x-2} & =3(x-2)+\frac{14(x-2)}{x-2} \\
7 x & =3 x-6+14 \\
4 x & =8 \\
x & =2
\end{aligned}
$$

Verify that no denominator is 0 to avoid division by 0 .
$\boldsymbol{x}-\mathbf{2}=\mathbf{2 - 2}=\mathbf{0}$ Thus $\boldsymbol{x}=\mathbf{2}$ is not a solution. This example has no solution.
3. Solve $\frac{7 x}{x-2}=7+\frac{16}{x-2}$

## Solution:

$$
\begin{aligned}
\frac{7 x}{x-2} & =7+\frac{16}{x-2} & & \\
\frac{7 x(x-2)}{x-2} & =7(x-2)+\frac{16(x-2)}{x-2} & & \text { distribute }(x-2) \\
7 x & =7 x-14+16 & & \text { reduce } \\
0 x & =-14+16 & & \text { subtract } 3 x \text { from both sides. } \\
0 & =2 & & \text { False statement cannot be corrected for any } \boldsymbol{x} .
\end{aligned}
$$

This example has no solution.
4. Solve $\frac{7 x}{x-2}=7+\frac{14}{x-2}$

## Solution:

$$
\begin{aligned}
\frac{7 x}{x-2} & =7+\frac{14}{x-2} & & \\
\frac{7 x(x-2)}{x-2} & =7(x-2)+\frac{14(x-2)}{x-2} & & \text { distribute }(x-2) \\
7 x & =7 x-14+14 & & \text { reduce } \\
0 x & =0 & & \text { subtract } 3 x \text { from both sides. } \\
0 & =0 & & \text { True statement cannot be falsified for any } \boldsymbol{x} .
\end{aligned}
$$

This example has infinitely many solutions. Any real number other than $\boldsymbol{x}=\mathbf{2}$ is a solution.
5. A car and a truck leave a park at the same time. The speed of the car is $\mathbf{4 0}$ miles per hour more than that of the truck. The truck's destination is $\mathbf{1 1 5}$ miles. The car's destination is $\mathbf{1 2 6}$ miles. The car reaches its destination in $\mathbf{3}$ hours less than the truck. What is the speed of the truck?

## Solution:

Let $\boldsymbol{x}$ be the speed of the truck. Then $\boldsymbol{x}+\mathbf{4 0}$ is the speed of the car.

|  | Rate | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Truck | $\boldsymbol{x}$ | $\boldsymbol{T}_{\boldsymbol{T}}$ | $\mathbf{1 1 5}$ |
| Car | $\boldsymbol{x}+\mathbf{4 0}$ | $\boldsymbol{T}_{\boldsymbol{C}}$ | $\mathbf{1 2 6}$ |

Since $\boldsymbol{R T}=\boldsymbol{D}, \boldsymbol{T}=\frac{\boldsymbol{D}}{\boldsymbol{R}}$
The Time the truck traveled was $\boldsymbol{T}_{\boldsymbol{T}}=\frac{\mathbf{1 1 5}}{\boldsymbol{x}}$
and
The Time the car traveled was $\boldsymbol{T}_{\boldsymbol{C}}=\frac{\mathbf{1 2 6}}{\boldsymbol{x}+\mathbf{4 0}}$
The car reaches its destination in $\mathbf{3}$ hours less than the truck, so

$$
\begin{aligned}
& T_{C}=T_{T}+3 \\
& \frac{115}{x}=\frac{126}{x+40}+3 \\
& \frac{115 x(x+40)}{x}=\frac{126 x(x+40)}{x+40}+3 x(x+40) \\
& 115(x+40)=126 x+3 x(x+40) \\
& 115 x+4600=126 x+3 x^{2}+120 x \\
& 3 x^{2}+131 x-4600=0 \\
& P=-13800 \\
& S=131 \\
& \begin{array}{l|l}
-1 & 13800
\end{array} \\
& \begin{array}{l|l}
-2 & 6900
\end{array} \\
& \begin{array}{l|l}
-3 & 4600
\end{array} \\
& \begin{array}{l|l}
-4 & 3450
\end{array} \\
& \begin{array}{l|l}
-5 & 2760
\end{array} \\
& \begin{array}{l|l}
-6 & 2300
\end{array} \\
& \begin{array}{l|l}
-8 & 1725
\end{array} \\
& \begin{array}{l|l}
-10 & 1380
\end{array} \\
& \begin{array}{l|l}
-60 \\
-69 & 230 \\
-200
\end{array} \quad \text { sum }=-69+200=131 \\
& 3 x^{2}+131 x-4600=0 \\
& 3 x^{2}-69 x+200 x-4600=0 \\
& 3 x(x-23)+200(x-23)=0 \\
& (x-23)(3 x+200)=0
\end{aligned}
$$

Thus $\boldsymbol{x}=-\frac{\mathbf{2 0 0}}{\mathbf{3}}$ which we reject because speed cannot be negative.
The speed of the truck is $\boldsymbol{x}=\mathbf{2 3}$ miles per hour.
6. A small pipe can fill a tank in $\mathbf{1 7}$ hours more than a large pipe. Together they fill the tank in $\mathbf{7 2}$ hours. How many hours does it take the big pipe to fill the tank by itself?

## Solution:

Let $\boldsymbol{x}$ be the number of hours it takes the large pipe to fill the tank. Then $\boldsymbol{x}+\mathbf{1 7}$ is the number of hours it takes the large pipe to fill the tank.

|  | Total time to fill the tank | Tank portion filled per hour |
| :---: | :---: | :---: |
| Large pipe | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| Small pipe | $\boldsymbol{x}+\mathbf{1 7}$ | $\frac{\mathbf{1}}{\boldsymbol{x}+\mathbf{1 7}}$ |
| Both pipes | $\mathbf{7 2}$ | $\frac{\mathbf{1}}{\mathbf{7 2}}$ |

The portion of the tank filled by the large pipe in $\mathbf{1}$ hour added to the portion of the tank filled by the small pipe in $\mathbf{1}$ hour equals the portion of the tank filled by both pipes in $\mathbf{1}$ hour. Thus

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{x+17} & =\frac{1}{72} \\
\frac{72 x(x+17)}{x}+\frac{72 x(x+17)}{x+17} & =\frac{72 x(x+17)}{72} \\
72(x+17)+72 x & =x(x+17) \\
72 x+1224+72 x & =x^{2}+17 x \\
0 & =x^{2}+17 x-144 x-1224 \\
0 & =x^{2}-127 x-1224
\end{aligned}
$$

$$
x^{2}+9 x-136 x-1224=0
$$

$$
x(x+9)-136(x+9)=0
$$

$$
(x+9)(x-136)=0
$$

Reject $\boldsymbol{x}=\mathbf{- 9}$ because a pipe cannot be on for a negative time.
Thus it takes $\mathbf{1 3 6}$ hours for the large pipe to fill the tank.

## Chapter 51

# Solving Equations with Proportions 

### 51.1 Youtube

https://www.youtube.com/playlist?list=PLD76F489A6DF27507\&feature=view_all

### 51.2 Basics

If $\frac{\boldsymbol{a}}{\boldsymbol{b}}=\frac{\boldsymbol{c}}{\boldsymbol{d}}$ then $\boldsymbol{a d}=\boldsymbol{b} \boldsymbol{c}$. Why?

$$
\begin{aligned}
\frac{a}{b} & =\frac{c}{d} \quad \text { multiply by } \mathrm{LCD}=b d \\
\frac{a b d}{b} & =\frac{c b d}{d} \\
a d & =b c
\end{aligned}
$$

### 51.3 Examples

Example 1:
Solve $\frac{x}{5}=\frac{23}{35}$

## Solution:

$$
\begin{array}{rll}
\frac{x}{5} & =\frac{23}{35} & \text { multiply both sides by } 5 \\
x & =\frac{5 \cdot 23}{35} & \text { Skip } 35 x=(5)(23) \text { followed by division by } 35 \\
x & =\frac{23}{7} & \text { reduce }
\end{array}
$$

Example 2:
On Monday 25 fish are caught in a small lake. They are tagged and then released. On Friday, 60 fish are caught. 12 of these fish were tagged. Estimate the total number of fish in the lake.

## Solution:

Let $\boldsymbol{x}$ be the number of fish in the lake. Assume that the proportion of $\mathbf{2 5}$ tagged fish released in the lake on Monday is to the total fish population $\boldsymbol{x}$ as the number of tagged fish caught on Friday (12) is to the total Friday catch (60).

$$
\begin{aligned}
\frac{25 \text { tagged }}{x \text { total }} & =\frac{12 \text { tagged }}{60 \text { total }} & & \\
12 x & =(25)(60) & & \text { postpone multiplication of }(\mathbf{2 5})(\mathbf{6 0 )} \\
x & =\frac{(\mathbf{2 5})(\mathbf{6 0})}{\mathbf{1 2}} & & \text { reduce } \\
x & =(\mathbf{2 5})(5)=\mathbf{1 2 5} & &
\end{aligned}
$$

We estimate that there are $\mathbf{1 2 5}$ fish in the lake.
Example 3:
Find the height $\boldsymbol{H}$ of a building that casts a $\mathbf{1 2 0} \mathrm{ft}$ shadow at the same time the shadow of a $\mathbf{6} \mathrm{ft}$ man is $\mathbf{8 f t}$ on level ground.


$$
\begin{aligned}
H & =\frac{6(120)}{8} \\
& =\frac{3 \cdot 2(4 \cdot 30)}{8}=90
\end{aligned}
$$

The building is $\mathbf{9 0} \mathrm{ft}$ tall.
Example 4:
Solve: $\frac{60}{x-2}=\frac{10}{x+3}$

## Solution:

$$
\begin{aligned}
\frac{60}{x-2} & =\frac{10}{x+3} \\
60(x+3) & =10(x-2) \\
60 x+180 & =10 x-20 \\
50 x & =-200 \\
x & =-\frac{200}{50}=-4
\end{aligned}
$$

Example 5:
$\mathbf{2 . 5}$ inches on a map correspond to a distance of $\mathbf{4 0}$ miles. What distance corresponds to $\mathbf{1 0 . 5}$ inches?

## Solution:

Let $\boldsymbol{x}$ be the distance in ft corresponding to $\mathbf{1 0 . 5}$ inches.
We equate two ratios of miles per inches, or inches per miles. We need to be consistent. It is imperative to keep the units.

$$
\begin{aligned}
\frac{40 \text { miles }}{2.5 \text { inches }} & =\frac{x \text { miles }}{10.5 \text { inches }} \\
2.5 x & =40(10.5) \\
x & =\frac{40(10.5)}{2.5} \\
x & =\frac{40(105)}{25} \\
x & =8(21)=168
\end{aligned}
$$

The distance corresponding to $\mathbf{1 0 . 5}$ inches is $\mathbf{1 6 8}$ miles.

### 51.4 Exercise 51

1. Solve $\frac{x}{6}=\frac{32}{54}$
2. On Monday 65 fish are caught in a small lake. They are tagged and then released. On Friday, 40 fish are caught. 10 of these fish are tagged. Estimate the total number of fish in the lake?
3. Find the height $\boldsymbol{H}$ of a building that casts a $\mathbf{9 0} \mathrm{ft}$ shadow at the same time the shadow of a $\mathbf{5} \mathrm{ft}$ man is $\mathbf{1 0} \mathrm{ft}$ on level ground.
4. Solve: $\frac{50}{x+4}=\frac{\mathbf{2 0}}{x-5}$

(man's shadow)
5. $\mathbf{3 . 5}$ inches on a map correspond to a distance of $\mathbf{7 0}$ miles. What distance in miles corresponds to 15.5 inches?

## STOP!

1. Solve $\frac{x}{6}=\frac{32}{54}$

## Solution:

$\frac{x}{6}=\frac{32}{54}$
multiply both sides by 6
$x=\frac{6 \cdot 32}{54}$ Skip $54 x=(6)(32)$ followed by division by 54
$\boldsymbol{x}=\frac{\mathbf{3 2}}{\mathbf{9}} \quad$ reduce
2. On Monday 65 fish are caught in a small lake. They are tagged and then released. On Friday, 40 fish are caught. 10 of these fish are tagged. Estimate the total number of fish in the lake?

## Solution:

Let $\boldsymbol{x}$ be the number of fish in the lake. Assume that the proportion of $\mathbf{6 5}$ tagged fish released in the lake on Monday is to the total fish population $\boldsymbol{x}$ as the number of tagged fish caught on Friday (10) is to the total Friday catch (40).

$$
\begin{array}{rlr}
\frac{65}{x} & =\frac{10}{40} & \\
10 x & =(65)(40) & \text { postpone multiplication of }(65)(40) \\
x & =\frac{(65)(40)}{10} \quad \text { reduce } \\
x & =(65)(4)=260 &
\end{array}
$$

We estimate that there are 260 fish in the lake.
3. Find the height $\boldsymbol{H}$ of a building casting a $\mathbf{9 0} \mathrm{ft}$ shadow when the shadow of a $\mathbf{5} \mathrm{ft}$ man is $\mathbf{1 0} \mathrm{ft}$.

## Solution:

$$
\begin{aligned}
\frac{H}{90} & =\frac{5}{10} \\
H & =\frac{5(90)}{10} \\
& =5(9)=45
\end{aligned}
$$



The building is 45 ft tall.
4. Solve: $\frac{\mathbf{5 0}}{\boldsymbol{x}+\mathbf{4}}=\frac{\mathbf{2 0}}{\boldsymbol{x}-\mathbf{5}}$

## Solution:

$$
\begin{aligned}
\frac{50}{x+4} & =\frac{20}{x-5} \\
50(x-5) & =20(x+4) \\
50 x-250 & =20 x+80 \\
30 x & =330 \\
x & =\frac{330}{30}=11
\end{aligned}
$$

5. $\mathbf{3 . 5}$ inches on a map correspond to a distance of $\mathbf{7 0}$ miles. What distance corresponds to $\mathbf{1 5 . 5}$ inches?

Solution:
Let $\boldsymbol{x}$ be the distance for $\mathbf{1 5 . 5}$ inches.
We equate two ratios of miles per inches, or inches per miles. We need to be consistent. It is imperative to keep the units.

$$
\begin{aligned}
\frac{70 \text { miles }}{3.5 \text { inches }} & =\frac{x \text { miles }}{15.5 \text { inches }} \\
3.5 x & =(70)(15.5) \\
x & =\frac{(70)(15.5)}{3.5} \\
x & =\frac{(70)(155)}{35} \\
x & =2(155)=310
\end{aligned}
$$

The distance corresponding to $\mathbf{1 5 . 5}$ inches is $\mathbf{3 1 0}$ miles.

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## Chapter 52

## Solving Systems of Equations by Graphing

### 52.1 Youtube

https://www.youtube.com/playlist?list=PL0C09609AA9B83B46\&feature=view_all

### 52.2 Basics

The method of solving a system of two (or more)equations in two unknowns graphically, with pencil and paper, is useful when verifying the reasonableness of a solution.

Using this method is not recommended for solving a system of two (or more) equations in two unknowns with pencil and paper if the solution has decimals.

The method is useful when approximating a solution with a graphing calculator. In this case, you may have to solve each equation for one variable $(\boldsymbol{y})$ in terms of the other $(\boldsymbol{x})$.

### 52.3 Examples

Example 1:
Graphically solve $\begin{cases}y & =x-5 \\ y & =-2 x+1\end{cases}$

## Solution:

$\boldsymbol{y}=\boldsymbol{x}-5$
$-5=0-5 \Rightarrow(0,-5)$
$-1=4-5 \Rightarrow(4,-1)$
$\boldsymbol{A}(\mathbf{0},-5)$ and $\boldsymbol{B}(4,-1)$ are
two points on the line $\boldsymbol{y}=\boldsymbol{x}-\mathbf{5}$.

$$
y=\quad-2 x+1
$$

$1=-2(0)+1 \Rightarrow(0,1)$
$5=-2(-2)+1 \Rightarrow(-2,5)$
$\boldsymbol{C}(\mathbf{0}, \mathbf{1})$ and $\boldsymbol{D}(-\mathbf{2}, \mathbf{5})$ are two points on the line $\boldsymbol{y}=-\mathbf{2 x}+\mathbf{1}$.
From the graph, the solution seems to be $\boldsymbol{E}(\mathbf{2}, \mathbf{- 3})$.
Since the system has one solution, it is said to be consistent.
Example 2:
Graphically solve $\left\{\begin{array}{lll}\boldsymbol{y} & =\boldsymbol{x}+\mathbf{1} \\ \boldsymbol{y} & =\boldsymbol{x}-\mathbf{1}\end{array}\right.$

## Solution:

| $y=x+1$ |
| :--- |
| $1=0+1$ |$\Rightarrow(0,1)$

$5=4+1 \Rightarrow(4,5)$
$\boldsymbol{A}(\mathbf{0}, \mathbf{1}), \boldsymbol{B}(4,5)$ are on $\boldsymbol{y}=\boldsymbol{x}+\mathbf{1}$.

| $y$ | $=$ |
| ---: | :--- |
| -1 | $=$ |
| $-1-1$ |  |$\Rightarrow(0,-1)$

$-3=-2-1 \Rightarrow(-2,-3)$
$\boldsymbol{C}(\mathbf{0},-\mathbf{1})$ and $\boldsymbol{D}(-\mathbf{2},-\mathbf{3})$ are two points on $\boldsymbol{y}=\boldsymbol{x}-\mathbf{1}$.
From the graph, there seems to be no solution since the lines seem to be parallel. Are the lines parallel? Yes, because both lines have the same slope $\boldsymbol{m}=\mathbf{1}$.

Since the system has no solution, it is said to be inconsistent.
Example 3:
Graphically solve $\begin{cases}y & =\pi x+1 \\ y & =3.14 x-1\end{cases}$

## Solution:

$\frac{y=\pi x+1}{1=\pi(0)+1} \Rightarrow(0,1)$
$\pi+1=\pi(1)+1 \Rightarrow(1,4.14159 \cdots)$
$\boldsymbol{A}(0,1)$ and $\boldsymbol{B}(1,4.14159 \cdots)$ are
two points on the line $\boldsymbol{y}=\boldsymbol{\pi} \boldsymbol{x}+\mathbf{1}$.

$$
\begin{array}{rll}
y & =3.14 x-1 & \\
\hline-4.14 & =3.14(0)-1 & \Rightarrow(0,-1) \\
-3 & =(3.14)(-1)-1 & \Rightarrow(-1,-4.14)
\end{array}
$$



Note: Each tick mark represents 1 unit.
$C(0,-1)$ and $\boldsymbol{D}(-1,-4.14)$ are two points on the line $y=3.14 x-1$.

From the graph, there seems to be no solution since the lines seem to be parallel. Are the lines parallel? No, because the slope of one line is $\boldsymbol{m}_{\mathbf{1}}=\boldsymbol{\pi} \approx \mathbf{3 . 1 4 1 5 9 \cdots}$ and the slope of the other line is $\boldsymbol{m}_{\mathbf{2}}=\mathbf{3 . 1 4}$. $m_{1} \neq \boldsymbol{m}_{2}$.

Since the system has a solution, it is said to be consistent.
Example 4:
Graphically solve $\left\{\begin{aligned} y & =x+1 \\ 2 y & =2 x+2\end{aligned}\right.$

## Solution:

| $y=x+1$ |
| :--- |
| $1=0+1$ |$\Rightarrow(0,1)$

$5=4+1 \Rightarrow(4,5)$
$\boldsymbol{A}(\mathbf{0}, \mathbf{1})$ and $\boldsymbol{B}(\mathbf{4}, \mathbf{5})$ are two points on the line $\boldsymbol{y}=\boldsymbol{x}+\mathbf{1}$.

| $2 y$ | $=$ | $2 x+2$ |
| ---: | ---: | ---: |
| 1 | $=$ | $0+1$ |$(0,1)$


$2(-1)=2(-2)+2 \Rightarrow(-2,-1)$
$\boldsymbol{C}(\mathbf{0},-\mathbf{1})$ and $\boldsymbol{D}(-\mathbf{2},-\mathbf{1})$ are two points on the line $\mathbf{2 y}=\mathbf{2 x}+\mathbf{2}$.
From the graph, there seem to be infinitely many solutions since the lines seem to coincide. Are the lines parallel? Yes, because both lines have the same slope $\boldsymbol{m}=\mathbf{1}$ and they have at least one point in common (like the $\boldsymbol{y}$-intercept $(\mathbf{0}, \mathbf{1})$ ).

This system of two equations in two unknowns is said to be dependent.

### 52.4 Exercise 52

1. Graphically solve $\left\{\begin{array}{l}y=2 x+10 \\ y=-3 x-5\end{array}\right.$
2. Graphically solve $\left\{\begin{array}{l}y=-2 x+3 \\ y=-2 x-3\end{array}\right.$
3. Graphically solve $\left\{\begin{array}{l}y=\sqrt{2} x+2 \\ y=1.414 x-2\end{array}\right.$
4. Graphically solve $\left\{\begin{aligned} y & =-x+1 \\ 3 y & =-3 x+3\end{aligned}\right.$

## STOP!

1. Graphically solve $\begin{cases}y & =2 x+10 \\ y & =-3 x-5\end{cases}$

## Solution:

| $y$ | $=$ | $2 x+10$ |
| ---: | ---: | ---: |
| 10 | $=$ | $2(0)+10$ |
| 0 | $=$ | $2(-5)+10 \Rightarrow(0,10)$ |
|  | $\Rightarrow 5,0)$ |  |

$\boldsymbol{A}(\mathbf{0}, \mathbf{1 0})$ and $\boldsymbol{B}(-\mathbf{5}, \mathbf{0})$ are two points on the line $\boldsymbol{y}=\mathbf{2 x}+\mathbf{1 0}$.

$$
\begin{array}{rlr}
y & = & -3 x-5 \\
\hline-5 & = & -3(0)-5 \\
1 & = & -3(-2)-5 \Rightarrow(0,-5) \\
& \Rightarrow(-2,1)
\end{array}
$$



Note: Each tick mark represents 1 unit.
$\boldsymbol{C}(\mathbf{0},-5)$ and $\boldsymbol{D}(-2,1)$ are two points on the line $\boldsymbol{y}=-\mathbf{3 x}-\mathbf{5}$.
From the graph, the solution seems to be $\boldsymbol{E}(-\mathbf{3}, \mathbf{4})$.
Since the system has one solution, it is consistent.
2. Graphically solve $\left\{\begin{array}{l}y=-2 x+3 \\ y=-2 x-3\end{array}\right.$

## Solution:

| $y=-2 x+3$ |
| :--- |
| $3=-2(0)+3$ |
| $1=-2(1)+3 \Rightarrow(0,3)$ |
| $1=1,1)$ |

$\boldsymbol{A}(\mathbf{0}, \mathbf{3})$ and $\boldsymbol{B}(\mathbf{1}, \mathbf{1})$ are two points on the line $\boldsymbol{y}=\mathbf{- 2 \boldsymbol { x }}+\mathbf{3}$.


Note: Each tick mark represents 1 unit.

$$
\begin{aligned}
& y= \\
&-3= \\
&-2(0)-3
\end{aligned} \Rightarrow(0,-3)
$$

$$
1=-2(-2)-3 \Rightarrow(-2,1)
$$

$\boldsymbol{C}(\mathbf{0},-\mathbf{3})$ and $\boldsymbol{D}(-\mathbf{2}, \mathbf{1})$ are two points on the line $\boldsymbol{y}=\mathbf{- 2 x}-\mathbf{3}$.
From the graph, there seems to be no solution since the lines seem to be parallel. Are the lines parallel? Yes, because both lines have the same slope $\boldsymbol{m}=\mathbf{- 2}$.
Since the system has no solution, it is inconsistent.
3. Graphically solve $\left\{\begin{array}{l}y=\sqrt{2} x+2 \\ y=1.414 x-2\end{array}\right.$

## Solution:

$y=\sqrt{2} x+2$
$2=\sqrt{2}(0)+2$$\Rightarrow(0,2)$
$3.414 \cdots=1.414 \cdots(1)+2 \Rightarrow(1,3.414 \cdots)$
$\boldsymbol{A}(0,2)$ and $\boldsymbol{B}(1,3.14159 \cdots)$ are two points on the line $\boldsymbol{y}=\sqrt{\mathbf{2}} \boldsymbol{x}+\mathbf{2}$.

$\begin{aligned} y & =1.414 x-2 \\ -2 & =1.414(0)-2\end{aligned} \Rightarrow(0,-2)$
Note: Each tick mark represents one unit.
$\boldsymbol{C}(\mathbf{0},-2)$ and $\boldsymbol{D}(-1,-3.414)$ are two points on the line $\boldsymbol{y}=1.414 \boldsymbol{x}-2$.
From the graph, there seems to be no solution since the lines seem to be parallel. Are the lines parallel? No, because the slope of one line is $\boldsymbol{m}_{1}=\sqrt{2} \approx 1.414213 \cdots$ and the slope of the other line is $\boldsymbol{m}_{\mathbf{2}}=1.414 . \boldsymbol{m}_{\mathbf{1}} \neq \boldsymbol{m}_{\mathbf{2}}$.
Since the system has a solution, it is said to be consistent.
4. Graphically solve $\left\{\begin{aligned} y & =-x+1 \\ 3 y & =-3 x+3\end{aligned}\right.$

## Solution:

$$
\begin{aligned}
y & =-x+1 \\
1 & =-0+1
\end{aligned} \Rightarrow(0,1)
$$

$\boldsymbol{A}(\mathbf{0}, \mathbf{1})$ and $\boldsymbol{B}(\mathbf{4},-\mathbf{3})$ are two points on the line $\boldsymbol{y}=-\boldsymbol{x}+\mathbf{1}$.


Note: Each tickmark represents one unit.

$$
\begin{array}{rlr}
3 y & = & -3 x+3 \\
\hline 3 & = & -3(0)+3 \\
3(3) & = & -3(-2)+3 \Rightarrow(0,1) \\
\Rightarrow(-2,-1)
\end{array}
$$

$\boldsymbol{C}(\mathbf{0}, \mathbf{1})$ and $\boldsymbol{D}(-2, \mathbf{3})$ are two points on the line $\mathbf{3 y}=-\mathbf{3 x}+\mathbf{3}$.
From the graph, there seem to be infinitely many solutions since the lines seem to coincide. Are the lines parallel? Yes, because both lines have the same slope $\boldsymbol{m}=\mathbf{- 1}$ and they have at least one point in common (like the $\boldsymbol{y}$-intercept $(\mathbf{0}, \mathbf{- 1})$ ).
This system of two equations in two unknowns is said to be dependent.

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## Chapter 53

## Solving Systems of Equations by Substitution

### 53.1 Youtube

https://www.youtube.com/playlist?list=PL87BABFDD91699612\&feature=view_all

### 53.2 Basics

The method of solving a system of two equations in two unknowns by substitution consists of replacing one variable by its equivalent in the other variable.

### 53.3 Examples

Example 1:
Use Substitution to solve
$\left\{\begin{array}{lr}(I) & y \\ (I I) & 4 x+y\end{array}\right) 2 x-7$
Solution:
(I) already shows $\boldsymbol{y}$ in terms of $\boldsymbol{x}$. Substitute (I) into (II).
(I)

$$
y=2 x-7
$$

(IIa) $4 x+(2 x-7)=23$

$$
\begin{aligned}
4 x+2 x-7 & =23 \\
6 x & =23+7 \\
6 x & =30 \text { or } x=\frac{30}{6}=5
\end{aligned}
$$

So $x=5$. Now $y=2(5)-7=3$
Using $\boldsymbol{y}$ from (I) was most efficient. Solve for and substitute $\boldsymbol{x}$ if you like fractions.
From $(I)$ get $(I a) 2 \boldsymbol{x}=\boldsymbol{y}+\mathbf{7}$ or $\boldsymbol{x}=\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{y}+7)$.
(Ia)

$$
\begin{aligned}
x & =\frac{1}{2}(y+7) \\
2(y+7)+y & =23 \\
2 y+14+y & =23 \\
3 y & =23-14 \\
3 y & =9 \\
y & =\frac{9}{3}=3
\end{aligned}
$$

So $y=3$. Now $x=\frac{1}{2}(3+7)=\frac{10}{2}=5$
Example 2:
Use Substitution to solve
$\left\{\begin{array}{lr}(I) & x-3 y\end{array} \quad=-14\right.$

## Solution:

Use equation $(\boldsymbol{I})$ to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get (Ia) $\boldsymbol{x}=\mathbf{3} \boldsymbol{y}-\mathbf{1 4}$.
Substitute (Ia) into (II).
(Ia)

$$
x=3 y-14
$$

(IIa) $5(3 y-14)-6 y=-16$
$15 y-70-6 y=-16$

$$
9 y=-16+70
$$

$$
9 y=54
$$

$$
y=\frac{54}{9}=6
$$

So $y=6$. Now $x=3(6)-14=18-14=4$.
Example 3:
Use Substitution to solve
$\left\{\begin{array}{lrl}(I) & 3 x-4 y & =-19 \\ (I I) & 6 x & =8 y-48\end{array}\right.$

## Solution:

Use equation (I) to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get
(Ia) $3 x=4 y-19$ or $x=\frac{1}{3}(4 y-19)$.
Substitute (Ia) into (II).

$$
\begin{aligned}
(\text { Ia }) & x \\
(I I a) & =\frac{1}{3}(4 y-19) \\
6\left(\frac{1}{3}(4 y-19)\right) & =8 y-48 \\
2(4 y-19) & =8 y-48 \\
8 y-38 & =8 y-48 \\
0 y & =-48+38 \\
0 & =-10
\end{aligned}
$$

$\mathbf{0}=-\mathbf{1 0}$ is false. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will correct this statement. This system of two equations in two variables has no solution.

Example 4:
Use Substitution to solve
$\left\{\begin{array}{lrl}(I) & 3 x-4 y & =-19 \\ (I I) & 6 x & =8 y-38\end{array}\right.$

## Solution:

Use equation ( $\boldsymbol{I})$ to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get
(Ia) $3 x=4 y-19$ or $x=\frac{1}{3}(4 y-19)$.
Substitute (Ia) into (II).
$(I a) \quad x=\frac{1}{3}(4 y-19)$
(IIa) $6\left(\frac{1}{3}(4 y-19)\right)=8 y-38$

$$
2(4 y-19)=8 y-38
$$

$$
8 y-38=8 y-38
$$

$$
0 y=-38+38
$$

$$
0=0
$$

$\mathbf{0}=\mathbf{0}$ is true. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will falsify this statement. This system of two equations in two variables has infinitely many solutions.

A word of caution here. Either $\boldsymbol{x}$ or $\boldsymbol{y}$ can be chosen arbitrarily. But the other variable depends on the choice of the first variable.

For example:
Let $y=a$ then $x=\frac{1}{3}(4 a-19)$.
If $y=a=4$ then $x=\frac{1}{3}(4(4)-19)=\frac{-3}{3}=-1$.
Check:
$\left\{\begin{array}{lrll}(I) & 3(-1)-4(4) & =-19 & \Rightarrow\end{array}-3-16=-19\right.$

### 53.4 Exercise 53

1. Use Substitution to solve
2. Use Substitution to solve

$$
\begin{cases}(I) & 2 x-3 y=-22 \\ (I I) & 5 x+6 y=53\end{cases}
$$

3. Use Substitution to solve

$$
\left\{\begin{array}{lr}
(I) & 5 x-6 y
\end{array}=-6012=18 y-12\right.
$$

4. Use Substitution to solve


## STOP!

1. Use Substitution to solve
$\left\{\begin{array}{lr}(I) & x\end{array}\right) 2 y-20$
Solution:
Equation $(\boldsymbol{I})$ already shows $\boldsymbol{x}$ in terms of $\boldsymbol{y}$. Substitute $(\boldsymbol{I})$ into (II).
(I)

$$
x=2 y-20
$$

(IIA) $3(2 y-20)+y=3$

$$
6 y-60+y=3
$$

$$
7 y=3+60
$$

$$
y=\frac{63}{7}=9
$$

So $\boldsymbol{y}=9$. Now $\boldsymbol{x}=2(9)-20=-2$
2. Use Substitution to solve
$\left\{\begin{array}{l}(I) \quad 2 x-3 y=-22 \\ (I I) \quad 5 x+6 y=53\end{array}\right.$

## Solution:

Use equation ( $\boldsymbol{I})$ to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get (Ia) $2 x=3 y-22$ or $x=\frac{1}{2}(3 y-22)$.
Substitute (Ia) into (II).
(Ia)

$$
x=\frac{1}{2}(3 y-22)
$$

$($ IIA $) \quad 5\left(\frac{1}{2}(3 y-22)\right)+6 y=53$

$$
5(3 y-22)+12 y=106
$$

$$
15 y-110+12 y=106
$$

$$
27 y=106+110
$$

$$
27 y=216
$$

$$
y=\frac{216}{27}=8
$$

So $\boldsymbol{y}=8$.

Now $x=\frac{1}{2}(3 \cdot 8-22)=\frac{24-22}{2}=1$.
3. Use Substitution to solve
$\left\{\begin{array}{lr}(I) & 5 x-6 y\end{array}=-6\right.$

## Solution:

Use equation $(\boldsymbol{I})$ to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get
$(I a) 5 x-6 y=-6$ or $x=\frac{1}{5}(6 y-6)$.
Substitute (Ia) into (II).
(Ia)

$$
x=\frac{1}{5}(6 y-6)
$$

(IIa) $15\left(\frac{1}{5}(6 y-6)\right)=18 y-12$

$$
\begin{aligned}
3(6 y-6) & =18 y-12 \\
18 y-18 & =18 y-12 \\
0 y-18 & =-12 \\
0 & =18-12=6
\end{aligned}
$$

$\mathbf{0}=\mathbf{6}$ is false. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will correct this statement. This system of two equations in two variables has no solution.
4. Use Substitution to solve
$\left\{\begin{array}{lrl}(I) & 5 x-6 y & =-6 \\ (I I) & 15 x & =18 y-18\end{array}\right.$

## Solution:

Use equation $(\boldsymbol{I})$ to solve for $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ to get
$(I a) 5 x-6 y=-6$ or $x=\frac{1}{5}(6 y-6)$.
Substitute (Ia) into (II).
(Ia)

$$
x=\frac{1}{5}(6 y-6)
$$

(IIa) $15\left(\frac{1}{5}(6 y-6)\right)=18 y-18$

$$
3(6 y-6)=18 y-18
$$

$$
18 y-18=18 y-18
$$

$$
0 y-18=-18
$$

$$
0=18-18=0
$$

$\mathbf{0}=\mathbf{0}$ is true. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will falsify this statement. This system of two equations in two variables has infinitely many solutions.
A word of caution here. Either $\boldsymbol{x}$ or $\boldsymbol{y}$ can be chosen arbitrarily. But the other variable depends on the choice of the first variable.

For example:
Let $y=a$ then $x=\frac{1}{5}(6 a-6)$.
If $y=a=6$ then $x=\frac{1}{5}(6(6)-6)=\frac{30}{5}=6$.
Check:
$\left\{\begin{array}{lrll}(I) & 5(6)-6(6) & =-6 & \Rightarrow-6=-6 \\ (I I) & 15(6) & =18(6)-18 & \Rightarrow 90=90\end{array}\right.$

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## Chapter 54

## Solving Systems of Equations by Elimination

### 54.1 Youtube

https://www.youtube.com/playlist?list=PL4E494E17822377C5\&feature=view_all

### 54.2 Basics

The method of solving a system of two equations in two unknowns by elimination (also called by addition) consists of eliminating one variable by adding the two equations.

We need opposite coefficients for the same variable in both equations.
Both sides of an equation can be multiplied by he same number.
The sum of two left sides of a system of two equations in two unknowns equals the sum of two right sides.
Example:
Suppose 7 (underlined) is to be eliminated from the following two equations.
Equation $I: 5+\mathbf{2} \cdot \underline{\mathbf{7}}=19$
Equation $I I: \mathbf{9}+\mathbf{3} \cdot \underline{\mathbf{7}}=\mathbf{3 0}$
To get opposite coefficients of $\underline{\mathbf{7}}$, multiply I by $\mathbf{3}$ and II by $\mathbf{- 2}$

$$
\begin{aligned}
3(I) \Rightarrow 3(5)+3(2 \cdot \underline{7}) & =3(19) \\
-2(I I) \Rightarrow-2(9)+(-2)(3 \cdot \underline{7}) & =-2(30)
\end{aligned}
$$

Rewrite:

$$
(I a) \Rightarrow 15+6 \cdot \underline{7}=57
$$

$$
(I I a) \Rightarrow-18+(-6) \cdot(\underline{7})=-60
$$

Add left sides and right sides: $\quad[\mathbf{1 5}+\mathbf{6} \cdot(\mathbf{7})]+[-\mathbf{1 8}+(-6)(\underline{\mathbf{7}})]=[57+(-60)]$

$$
\begin{aligned}
{[15]+[-18]+0 \cdot \underline{7} } & =-3 \\
-3 & =-3
\end{aligned}
$$

We succeeded in eliminating $\underline{\mathbf{7}}$.

### 54.3 Examples

## Example 1:

Use elimination to solve
$\left\{\begin{array}{lr}(I) & y=2 x-7 \\ (I I) & 4 x+y\end{array}\right.$

## Solution:

Method 1: Eliminate $\boldsymbol{y}$.
$\left\{\begin{array}{lr}(I) & y=2 x-7 \\ (I I) & 4 x+y\end{array}\right)=23$
$\left\{\begin{array}{rl}(\boldsymbol{I} \boldsymbol{a}) & -\mathbf{2 x}+\boldsymbol{y}\end{array}=-\mathbf{7}\right.$ Get both variables on the same (left) side.
$\left\{\begin{array}{l}(I b) \quad 2 x-y=7 \\ (I I) \quad 4 x+y=23\end{array}\right.$ multiply by -1.

$$
\begin{array}{rlrl}
2 x-y+4 x+y & =7+23 & & \text { Add both sides separately. } \\
2 x+4 x & =30 & y & y \text { has been eliminated. } \\
6 x & =30 & & \\
x & =5 & &
\end{array}
$$

We also need to solve for $\boldsymbol{y}$.
So $x=5$. Now $y=2(5)-7=3$
Method 2: Eliminate $\boldsymbol{x}$.
$\left\{\begin{array}{lrl}(I) & y & =2 x-7 \\ (I I) & 4 x+y & =23\end{array}\right.$

```
\(\left\{\begin{aligned}(\text { Ia }) & -2 x+y\end{aligned}\right)=-7 \quad\) Get both variables on the same (left) side.
\(\left\{\begin{aligned}(I c)-4 x+2 y & =-14 \text { multiply by } 2 . \\ (I I) & 4 x+y\end{aligned}\right)\)
    \(-4 x+2 y+4 x+y=-14+23\)
    \(2 y+y=9\)
    \(3 y=9 \quad\) or \(\boldsymbol{y}=\mathbf{3}\)
```

We also need to solve for $\boldsymbol{x}$.
So $y=3$. Now pick either original equation. (I) $\mathbf{3}=\mathbf{2 x - 7}$ or $\mathbf{2 x}=\mathbf{1 0}$ thus $\boldsymbol{x}=\mathbf{5}$.
Example 2:
Use elimination to solve
$\left\{\begin{array}{l}(I) \quad x-3 y=-14 \\ (I I) \quad 5 x-6 y=-16\end{array}\right.$

## Solution:

$\left\{\begin{array}{lr}(I) & x-3 y\end{array}=-14\right.$
$\left\{\begin{array}{rl}(\text { Ia }) & -2 x+6 y\end{array}=28 \quad\right.$ multiply both sides by -2
Add both sides. Note the elimination of $\boldsymbol{y}$.

$$
\begin{aligned}
-2 x+6 y+5 x-6 y & =28-16 \\
3 x & =12 \\
x & =4
\end{aligned}
$$

From (I) $4-3 y=-14,-3 y=-18$ or $y=\frac{-18}{-3}=6$
Example 3:
Use elimination to solve
$\left\{\begin{array}{lrl}(I) & 3 x-4 y & =-19 \\ (I I) & 6 x & =8 y-48\end{array}\right.$

## Solution:

$\left\{\begin{array}{ll}(\text { I }) & 3 x-4 y=-19 \\ (\text { IIa) } & 6 x-8 y=-48\end{array}\right.$ subtract $8 y$ from both sides.
$\left\{\begin{array}{rl}(I) & 3 x-4 y\end{array}=-19 \quad\right.$ divide both sides by $-\mathbf{2}$
Now add both sides. The $\boldsymbol{y}$ s are eliminated (and so are the $\boldsymbol{x s}$ ).
$3 x-4 y+(-3 x)+4 y=-19+24$ or $0=5$.
$\mathbf{0}=\mathbf{5}$ is false. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will correct this statement. This system of two equations in two variables has no solution.

Example 4:
Use elimination to solve


## Solution:

$\left\{\begin{array}{ll}(I) & 3 x-4 y=-19 \\ (I I a) & 6 x-8 y=-38\end{array}\right.$ subtract $8 y$ from both sides.
$\left\{\begin{array}{rl}(I) & 3 x-4 y\end{array}=-19 \quad\right.$ divide both sides by -2
Now add both sides. The $\boldsymbol{y}$ s are eliminated (and so are the $\boldsymbol{x s}$ ).
$3 x-4 y+-3 x+4 y=-19+19$ or $0=0$.
$\mathbf{0}=\mathbf{0}$ is true. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will falsify this statement. This system of two equations in two variables has infinitely many solutions.

A word of caution here. Either $\boldsymbol{x}$ or $\boldsymbol{y}$ can be chosen arbitrarily. But the other variable depends on the choice of the first variable.

For example:
Let $y=a$ then $x=\frac{1}{\mathbf{3}}(4 a-19)$.
If $y=a=4$ then $x=\frac{1}{3}(4 \cdot 4-19)=\frac{-3}{3}=-1$.
Check:
$\left\{\begin{array}{lrll}(I) & 3(-1)-4(4) & =-19 & \Rightarrow\end{array}-3-16=-19\right.$

### 54.4 Exercise 54

1. Use elimination to solve

$$
\left.\left\{\begin{array}{lr}
(I) & x
\end{array}\right)=2 y-20\right\}
$$

2. Use elimination to solve
$\left\{\begin{array}{ll}(I) & 2 x-3 y\end{array}=-22\right.$
3. Use elimination to solve
$\left\{\begin{array}{lr}(I) & 5 x-6 y\end{array}=-6012\right.$
4. Use elimination to solve


## STOP!

1. Use elimination to solve
$\left\{\begin{array}{lr}(I) & x\end{array}\right)=2 y-20$
Solution:
$\left\{\begin{array}{l}(I a) \quad x-2 y=-20 \text { subtract } y \text { from both sides. } \\ (I I) \quad 3 x+y=3\end{array}\right.$
$\left\{\begin{array}{ll}(I a) & x-2 y \\ = & -20 \\ (I I a) & 6 x+2 y\end{array}=6 \quad\right.$ multiply by 2.
Add both sides.
$x-2 y+6 x+2 y=-20+6$ or $\mathbf{7 x}=-14$ from which we see $\boldsymbol{x}=-2$.
Now (II) $\mathbf{3}(-2)+\boldsymbol{y}=\mathbf{3}$ leads to $\boldsymbol{y}=\mathbf{9}$.
2. Use elimination to solve
$\left\{\begin{array}{ll}(I) & 2 x-3 y\end{array}=-22\right.$.
Solution:
$\left\{\begin{array}{l}(I) \quad 2 x-3 y=-22 \\ (I I) \quad 5 x+6 y=53\end{array}\right.$
$\left\{\begin{array}{l}(I a) \quad 4 x-6 y=-44 \text { multiply by } 2 \\ (I I) \quad 5 x+6 y=53\end{array}\right.$
Add both sides.
$4 x-6 y+5 x+6 y=-44+53$ or $4 x+5 x=9$ from which we see that $9 x=9$, Thus $x=1$.
3. Use elimination to solve
$\left\{\begin{array}{lr}(I) & 5 x-6 y\end{array}=-6012\right.$
Solution:
$\left\{\begin{array}{lrl}(I) & 5 x-6 y & =-6 \\ (I I a) & 15 x-18 y & =-12 \quad \text { subtract } 18 y \text { from both sides. }\end{array}\right.$
$\left\{\begin{array}{ll}(\text { Ia }) \quad-15 x+18 y & =18 \\ (\text { IIa }) & 15 x-18 y\end{array}=-12\right.$ multiply both sides by -3.
$-15 x+18 y+15 x-18 y=18-12$ or $0=6$
$\mathbf{0}=\mathbf{6}$ is false. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will correct this statement. This system of two equations in two variables has no solution.
4. Use elimination to solve
$\left\{\begin{array}{lr}(I) & 5 x-6 y\end{array}=-6018\right.$ (II) $15 x=18 y-18$
Solution:
$\left\{\begin{array}{rl}(I) & 5 x-6 y\end{array}=-6.6\right.$ subtract $18 y$ from both sides.
$\left\{\begin{array}{rl}(I) & 5 x-6 y\end{array}=-6.1\right.$ divide both sides by $-\mathbf{3}$.
Add both sides.
$5 x-6 y+(-5 x)+6 y=-6+6$ so $0=0$
$\mathbf{0}=\mathbf{0}$ is true. No value of $\boldsymbol{y}$ (or $\boldsymbol{x}$ ) will falsify this statement. This system of two equations in two variables has infinitely many solutions.
A word of caution here. Either $\boldsymbol{x}$ or $\boldsymbol{y}$ can be chosen arbitrarily. But the other variable depends on the choice of the first variable.
For example:
Let $y=a$ then $x=\frac{\mathbf{1}}{\mathbf{5}}(\mathbf{6 a - 6})$.
If $y=a=6$ then $x=\frac{1}{5}(6 \cdot 6-6)=\frac{30}{5}=6$.
Check:
$\left\{\begin{array}{lrll}(I) & 5(6)-6(6) & =-6 & \Rightarrow-6=-6 \\ (I I) & 15(6) & =18(6)-18 & \Rightarrow 90=90\end{array}\right.$

## Chapter 55

## Solving Systems Application Problems Using Systems of Equations

### 55.1 Youtube

https://www.youtube.com/playlist?list=PL98353A3C43F9B589\&feature=view_all

### 55.2 Basics

You have seen most of these problems before. So what is new? Maybe nothing.
If you are asked to solve for two quantities, a condition is given connecting these two quantities.
For example: How many gallons of a $\mathbf{1 0} \%$ alcohol solution must be added to a $\mathbf{6 0} \%$ alcohol solution to get 100 gallons of ....

We used to say
let $\boldsymbol{x}$ be the number of gallons of the $\mathbf{1 0} \%$ solution, then $\mathbf{1 0 0}-\boldsymbol{x}$ is the number of gallons of the $\mathbf{6 0} \%$ solution

Now we say
let $\boldsymbol{x}$ be the number of gallons of the $\mathbf{1 0} \%$ solution
and
$\boldsymbol{y}$ is the number of gallons of the $\mathbf{6 0 \%}$ solution
but we add the equation $\boldsymbol{x}+\boldsymbol{y}=\mathbf{1 0 0}$. This new equation is solved for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ so that $\boldsymbol{y}=\mathbf{1 0 0}-\boldsymbol{x}$ which is substituted in the other equation.

Then proceed exactly like you did with just one variable.
Example 1:
A jet flying with the wind travels $\mathbf{2 , 1 2 0}$ miles in $\mathbf{4}$ hours. Against the same wind it can cover 2, $\mathbf{8 0 0}$ miles in $\mathbf{6}$ hours and 40 minutes. What is the speed of the jet in calm air and what is the speed of the wind?

## Solution:

Let $\boldsymbol{x}$ be the speed of the jet and $\boldsymbol{y}$ the speed of wind.

|  | Rate | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Flying with the wind | $x+y$ | $\mathbf{4}$ | $4(\boldsymbol{x}+\boldsymbol{y})$ |
| Flying against the wind | $\boldsymbol{x}-\boldsymbol{y}$ | $\mathbf{6} \frac{\mathbf{2}}{\mathbf{3}}$ | $\frac{\mathbf{2 0}}{\mathbf{3}}(\boldsymbol{x}-\boldsymbol{y})$ |

The system of two equations in two unknowns is
$\left\{\begin{aligned}(I) \quad 4(x+y) & =2120 \quad \text { divide by } 4 \\ (I I) & \frac{20}{3}(x-y)\end{aligned}\right) 2800 \quad$ ( $\quad=$
$\left\{\begin{array}{l}(\text { Ia }) \quad x+y=530 \\ (\text { IIa }) \quad x-y=\frac{3}{20} 2800 \quad \text { multiply both sides by by } \frac{20}{3}\end{array}\right.$
$\left\{\begin{array}{l}(I a) \quad x+y=530 \\ (I I b) \quad x-y=\mathbf{3 ( 1 4 0 )} \text { reduce }\end{array}\right.$
$2 x=530+420$ or $2 x=950$ thus $x=475$.
From (Ia) $475+\boldsymbol{y}=530$ or $\boldsymbol{y}=530-475=55$.
The jet flies at $\mathbf{4 7 5} \mathrm{mph}$ in calm air. The speed of the wind is $\mathbf{5 5} \mathrm{mph}$.

## Example 2:

It takes crew A 18 hours longer to fill a depression with sand than crew B. If the two crews work together, the filling would take 40 hours. How long would each crew take working individually?

## Solution:

Let $\boldsymbol{x}$ be the number of hours it takes crew A to fill the depression, $\boldsymbol{y}$ be the number of hours it takes crew B.

|  | Total time | Portion of job in 1 hour |
| :---: | :---: | :---: |
| crew A | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| crew B | $\boldsymbol{y}$ | $\frac{\mathbf{1}}{\boldsymbol{y}}$ |
| both crews | $\mathbf{4 0}$ | $\frac{\mathbf{1}}{\mathbf{4 0}}$ |

It takes crew A 18 hours longer than crew B, so $\boldsymbol{x}=\boldsymbol{y}+\mathbf{1 8}$ or $\boldsymbol{y}=\boldsymbol{x}-\mathbf{1 8}$.
Portion of job done by crew A in one hour plus portion of job done by crew B in one hour equals portion of job done by both crews in one hour.

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{y} & =\frac{1}{40} \\
\frac{1}{x}+\frac{1}{x-18} & =\frac{1}{40} \\
\frac{40 x(x-18)}{x}+\frac{40 x(x-18)}{x-18} & =\frac{40 x(x-18)}{40} \\
40(x-18)+40 x & =x(x-18) \\
40 x-720+40 x & =x^{2}-18 x \\
80 x-720 & =x^{2}-18 x \\
0 & =x^{2}-98 x+720
\end{aligned}
$$

$$
\quad \text { sum }=-8-90=-98
$$

$$
\begin{aligned}
x^{2}-98 x+720 & =0 \\
x^{2}-90 x-8 x+720 & =0 \\
x(x-90)-8(x-90) & =0 \\
(x-90)(x-8) & =0
\end{aligned}
$$

$\boldsymbol{x}=\mathbf{8}$ defies logic because one crew working alone cannot outperform both crews working together.
It takes crew A 90 hours to fill the depression working alone.

It takes crew B 90-18 = 72 hours to fill the depression working alone.
Example 3:
How many gallons of a $\mathbf{1 2} \%$ acid solution must be added to a $\mathbf{7 6} \%$ acid solution to get $\mathbf{8 0}$ gallons of a $\mathbf{4 8} \%$ acid solution?

## Solution:

Let $\boldsymbol{x}$ be the number of gallons of $\mathbf{1 2} \%$ acid solution and $\boldsymbol{y}$ the number of gallons of $\mathbf{7 6} \%$ acid solution.

| concentration | amount | acid |
| :---: | :---: | :---: |
| $12 \%$ solution | $\boldsymbol{x}$ | $\mathbf{0 . 1 2 \boldsymbol { x }}$ |
| $76 \%$ solution | $\boldsymbol{y}$ | $\mathbf{0 . 7 6 \boldsymbol { y }}$ |
| $48 \%$ solution | $\mathbf{8 0}$ | $\mathbf{0 . 4 8 ( 8 0 )}$ |



$$
\left\{\begin{array}{rl}
(I) & x+y
\end{array}=80 \quad\left\{\begin{aligned}
(I I) & 0.12 x+0.76 y
\end{aligned}\right)=0.48(80)\right.
$$

$$
12 x+76(80-x)=3840
$$

$$
12 x+6080-76 x=3840
$$

$$
-64 x=3840-6080
$$

$$
-64 x=-2240
$$

$$
x=\frac{2240}{64}=35
$$

Since $x=35, y=80-35=45$.
Add $\mathbf{3 5}$ gallons of $\mathbf{1 2 \%}$ acid solution to $\mathbf{4 5}$ gallons of $\mathbf{7 6 \%}$ acid solution to get $\mathbf{8 0}$ gallons of $\mathbf{4 8} \%$ acid solution.

Example 4:
Once upon a time when movie tickets were cheaper, regular admission was $\$ 4.25$ per ticket and $\$ 3.50$ for senior citizens. There were $\mathbf{1 7 0}$ less senior tickets sold than regular tickets. How many tickets of each category were sold if regular admission exceeded senior citizen admission by $\$ \mathbf{7 6 9}$ ?

## Solution:

Let $\boldsymbol{x}$ be the number of regular tickets and $\boldsymbol{y}$ the number of senior citizen tickets.

| Category | Number of tickets | Price per ticket | total revenue |
| :---: | :---: | :---: | :---: |
| Regular | $\boldsymbol{x}$ | $\mathbf{4 . 2 5}$ | $\mathbf{4 . 2 5 x}$ |
| Senior | $\boldsymbol{y}$ | $\mathbf{3 . 5 0}$ | $\mathbf{3 . 5 0} \boldsymbol{y}$ |

There were $\mathbf{1 7 0}$ less senior tickets sold than regular tickets, so $\boldsymbol{y}=\boldsymbol{x} \mathbf{- 1 7 0}$.
Regular admission exceeded senior citizen admission by $\$ \mathbf{7 6 9}$ means that $\mathbf{4 . 2 5 x}-\mathbf{3 . 5 0 y}=\mathbf{7 6 9}$.
$\left\{\begin{array}{lrl}(I) & 4.25 x-3.50 y & =769 \\ (I I) & y & =x-170\end{array}\right.$

$$
\begin{aligned}
4.25 x-3.50(x-170) & =769 \\
4.25 x-3.50 x+595 & =769 \\
0.75 x & =769-595 \\
0.75 x & =174 \\
x & =\frac{174}{0.75}=232
\end{aligned}
$$

Thus there were $\mathbf{2 3 2}$ regular tickets and $\mathbf{2 3 2} \mathbf{- 1 7 0}=\mathbf{6 2}$ senior citizen tickets.

### 55.3 Exercise 55

1. A jet flying with the wind can travel 996 miles in $\mathbf{4}$ hours. Against the same wind speed it can cover $\mathbf{1 , 0 8 5}$ miles in $\mathbf{5}$ hours. What is the speed of the jet in calm air and what is the speed of the wind?
2. It takes crew A 32 hours longer to fill a depression with sand than crew B. If the two crews were together, the filling would take $\mathbf{3 0}$ hours. How long would each crew take working individually?
3. How many gallons of a $\mathbf{1 8} \%$ acid solution must be added to a $\mathbf{6 8} \%$ acid solution to get 100 gallons of a $\mathbf{5 8} \%$ acid solution?
4. Once upon a time when concert tickets were cheaper, regular admission was $\$ \mathbf{1 9 . 7 5}$ per ticket and $\$ 12.25$ for senior citizens. There were $\mathbf{3 6 4}$ less senior tickets sold than regular tickets. How many tickets of each category were sold if regular admission exceeded senior citizen admission by $\mathbf{\$ 1 0}, \mathbf{3 3 9}$ ?

## STOP!

1. A jet flying with the wind can travel 996 miles in $\mathbf{4}$ hours. Against the same wind speed it can cover $\mathbf{1}, \mathbf{0 8 5}$ miles in $\mathbf{5}$ hours. What is the speed of the jet in calm air and what is the speed of the wind?

## Solution:

Let $\boldsymbol{x}$ be the speed of the jet and $\boldsymbol{y}$ the speed of wind.

|  | Rate | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Flying with the wind | $\boldsymbol{x}+\boldsymbol{y}$ | $\mathbf{4}$ | $\mathbf{4 ( x + y )}$ |
| Flying against the wind | $\boldsymbol{x}-\boldsymbol{y}$ | $\mathbf{5}$ | $\mathbf{5}(\boldsymbol{x}-\boldsymbol{y})$ |

The system of two equations in two unknowns is
$\left\{\begin{array}{lll}(I) & 4(x+y)=996 & \text { divide by } 4 \\ (I I) & 5(x-y)=1,085 & \text { divide by } 5\end{array}\right.$
$\begin{cases}(I a) & x+y=249 \\ (I I a) & x-y=217\end{cases}$
$2 x=249+217$ or $2 x=466$ thus $x=233$.
From (Ia) $233+\boldsymbol{y}=249$ or $\boldsymbol{y}=249-233=16$.
The jet flies at $\mathbf{2 3 3} \mathrm{mph}$ in calm air. The speed of the wind is $\mathbf{1 6} \mathbf{m p h}$.
2. It takes crew A 32 hours more to fill a depression than crew B. If the two crews work together, the filling takes $\mathbf{3 0}$ hours. How long does each crew take working individually?

## Solution:

Let $\boldsymbol{x}$ be the number of hours it takes crew A to fill the depression and $\boldsymbol{y}$ be the number of hours it takes crew B.

|  | Total time | Portion of job in 1 hour |
| :---: | :---: | :---: |
| crew A | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| crew B | $\boldsymbol{y}$ | $\frac{\mathbf{1}}{\boldsymbol{y}}$ |
| both crews | $\mathbf{3 0}$ | $\frac{\mathbf{1}}{\mathbf{3 0}}$ |

Crew A takes $\mathbf{3 2}$ hours more than B , so $\boldsymbol{x}=\boldsymbol{y}+\mathbf{3 2}$ or $\boldsymbol{y}=\boldsymbol{x}-\mathbf{3 2}$.
Crew A's job portion per hour plus crew B's portion done per hour $=$ job portion done by both crews in 1 hour.

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{y} & =\frac{1}{30} \\
\frac{1}{x}+\frac{1}{x-32} & =\frac{1}{30} \\
\frac{30 x(x-32)}{x}+\frac{30 x(x-32)}{x-32} & =\frac{30 x(x-32)}{30} \\
30(x-32)+30 x & =x(x-32) \\
30 x-960+30 x & =x^{2}-32 x \\
60 x-960 & =x^{2}-32 x \\
0 & =x^{2}-92 x+960
\end{aligned}
$$

$$
\begin{aligned}
& P=960 \\
& S=-92 \\
& \begin{array}{r|l}
-1 & -960 \\
-2 & -480 \\
-3 & -320 \\
-4 & -240 \\
-5 & -192 \\
-6 & -160 \\
-8 & -120 \\
-10 & -96 \\
-12 & -80
\end{array} \quad \text { sum }=-12-80=-92 \\
& x^{2}-92 x+960=0 \\
& x^{2}-12 x-80 x+960=0 \\
& x(x-12)-80(x-12)=0 \\
& (x-12)(x-80)=0
\end{aligned}
$$

$\boldsymbol{x}=12$ defies logic because one crew working alone cannot outperform both crews working together.
It takes crew A $\mathbf{8 0}$ hours to fill the depression working alone.
It takes crew B 80-32=48 hours to fill the depression working alone.
3. solution How many gallons of $\mathbf{1 8} \%$ acid solution are added to $\mathbf{6 8} \%$ to get 100 gallons of $\mathbf{5 8} \%$ acid solution?

## Solution:

Let $\boldsymbol{x}$ be the number of gallons of $18 \%$ acid solution and $\boldsymbol{y}$ the number of gallons of $68 \%$ acid solution.

| concentration | amount | acid |
| :---: | :---: | :---: |
| $18 \%$ solution | $\boldsymbol{x}$ | $\mathbf{0 . 1 8 \boldsymbol { x }}$ |
| $68 \%$ solution | $\boldsymbol{y}$ | $\mathbf{0 . 6 8} \boldsymbol{y}$ |
| $58 \%$ solution | $\mathbf{1 0 0}$ | $\mathbf{0 . 5 8 ( 1 0 0 )}$ |



$$
\left.\begin{array}{l}
\left\{\begin{array}{lr}
(I) & x+y \\
(I I) & 0.18 x+0.68 y
\end{array}=0.58(100)\right.
\end{array}\right\} \begin{aligned}
& (I a) \\
& \left\{\begin{array}{lrl}
(I I a & 18 x+68 y= & 58(10)
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
18 x+68(100-x) & =5,800 \\
18 x+6800-68 x & =5800 \\
-50 x & =-6800+5800 \\
-50 x & =-1000 \\
x & =\frac{1000}{50}=20
\end{aligned}
$$

Since $x=20, y=100-20=80$.
Add $\mathbf{2 0}$ gallons of $\mathbf{1 8} \%$ acid solution to $\mathbf{8 0}$ gallons of $\mathbf{6 8 \%}$ acid solution to get $\mathbf{1 0 0}$ gallons of $\mathbf{5 8} \%$ acid solution.
4. Once upon a time when concert tickets were cheaper, regular admission was $\$ \mathbf{1 9 . 7 5}$ per ticket and $\$ 12.25$ for senior citizens. There were $\mathbf{3 6 4}$ less senior tickets sold than regular tickets. How many tickets of each category were sold if regular admission exceeded senior citizen admission by $\mathbf{\$ 1 0}, \mathbf{3 3 9}$ ?

## Solution:

Let $\boldsymbol{x}$ be the number of regular tickets and $\boldsymbol{y}$ the number of senior citizen tickets.

| Category | Number of tickets | Price per ticket | total revenue |
| :---: | :---: | :---: | :---: |
| Regular | $\boldsymbol{x}$ | $\mathbf{1 9 . 7 5}$ | $\mathbf{1 9 . 7 5 \boldsymbol { x }}$ |
| Senior | $\boldsymbol{y}$ | $\mathbf{1 2 . 2 5}$ | $\mathbf{1 2 . 2 5} \boldsymbol{y}$ |

There were $\mathbf{3 6 4}$ less senior tickets sold than regular tickets, so $\boldsymbol{y}=\boldsymbol{x} \mathbf{- 3 6 4}$.
Regular admission exceeded senior citizen admission by $\$ \mathbf{1 0}, \mathbf{3 3 9}$ means that $19.75 x-12.25 y=10,339$.

$$
\begin{aligned}
(I) \quad 19.75 x-12.25 y & =10,339 \\
y & =x-364 \\
(I I) 10 & =10,339 \\
19.75 x-12.25 y & =10,339 \\
19.75 x-12.25(x-364) & =10,339 \\
19.75 x-12.25 x+4459 & =10,339-4459 \\
7.5 x & =10 \\
7.5 x & =5880 \\
x & =\frac{5880}{7.5}=784
\end{aligned}
$$

Thus there were $\mathbf{7 8 4}$ regular tickets and $\mathbf{7 8 4 - 3 6 4}=\mathbf{4 2 0}$ senior citizen tickets.

## Chapter 56

## Linear Inequalities in Two Variables

### 56.1 Youtube

https://www.youtube.com/playlist?list=PL312D3684F98313B7\&feature=view_all

### 56.2 Basics

Given $2 x-3 y \leq 6$ there are a lot of points whose coordinates satisfy the given inequality and a lot of points that don't.

Just for kicks let's try $\boldsymbol{A}(2,3),, B(-2,3), C(-2,-3), D(2,-3)$.
For $A(2,3) \quad 2(2)-3(3) \leq 6$ or $4-9 \leq 6$ is true.
For $B(-2,3) \quad 2(-2)-3(3) \leq 6$ or $-4-9 \leq 6$ is true.
For $C(-2,-3) \quad 2(-2)-3(-3) \leq 6$ or $-4+9 \leq 6$ is true.
For $D(2,-3) \quad 2(2)-3(-3) \leq 6$ or $4+9 \leq 6$ is false.
For the given inequality, the line $\mathbf{2 x}-\mathbf{3 y}=\mathbf{6}$ subdivides the Cartesian plane into two regions. All the points in one region have points whose coordinates satisfy the inequality. All the points in the other region do not.

After graphing the subdividing boundary, using one test point off the boundary in either region will show which region has the desired points whose coordinates satisfy the inequality.

Example 1:
Shade the solution set of $\mathbf{2 x}-\mathbf{3} \boldsymbol{y} \leq \mathbf{6}$.

## Solution:



| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\mathbf{2 x}-\mathbf{3 y = 6}$ | the boundary |
| :--- | :--- | :--- | :--- |
| $\mathbf{0}$ | $-\mathbf{2}$ | $\mathbf{2 ( 0 ) - 3 ( - 2 ) = 6}$ | $\boldsymbol{y}$ intercept. |
| $\mathbf{3}$ | $\mathbf{0}$ | $\mathbf{2 ( 3 ) - 3 ( 0 ) = 6}$ | $\boldsymbol{x}$ intercept. |

Easiest point to test: $(\mathbf{0}, \mathbf{0})$ (if not on the boundary.)
$2(0)-3(0) \leq 6$ or $0 \leq 6$ is true.
Shade the region containing ( $\mathbf{0}, \mathbf{0}$ ).
Example 2:
Shade the solution set of $\boldsymbol{x}+\boldsymbol{y}>\mathbf{0}$.

## Solution:



| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{x}+\boldsymbol{y}=\mathbf{0}$ | the boundary |
| :--- | :--- | :--- | :--- |
| $\mathbf{0}$ | $\mathbf{0}$ | $(\mathbf{0})+(\mathbf{0})=\mathbf{0} \quad \boldsymbol{y}$ (and $-\boldsymbol{x})$ intercept. |  |
| $\mathbf{3}$ | $-\mathbf{3}$ | $(\mathbf{3})+(-\mathbf{3})=\mathbf{0}$ |  |

Don't use ( $\mathbf{0}, \mathbf{0}$ ) as test point. It's on the boundary.)
Use a point on one of the axes like $(\mathbf{0}, \mathbf{3})$.
$\boldsymbol{x}+\boldsymbol{y}<\mathbf{0}$ becomes $\mathbf{0}+\mathbf{3}>\mathbf{0}$ which is true,
so shade the region containing the test point.
Points on the boundary don't satisfy this inequality.
$=$ is not part of $\boldsymbol{x}+\boldsymbol{y}>\boldsymbol{0}$. The boundary is dashed.
Example 3:
Shade the solution set of $\left\{\begin{array}{l}\boldsymbol{y} \leq \mathbf{- 2} \\ \boldsymbol{y} \geq \mathbf{2}\end{array}\right.$ or
Solution:


The shaded region is the solution set for one inequality or the other or both. The keyword is OR, not and. If the problem had stated "shade the solution set for $\boldsymbol{y} \leq \mathbf{- 2}$ AND $\boldsymbol{y} \geq \mathbf{2}$ " then there would be no solution since no point can be above $\boldsymbol{y} \geq \mathbf{2}$ AND below $\boldsymbol{y} \leq \mathbf{- 2}$ simultaneously.

### 56.3 Exercise 56

1. Shade the solution set of $\mathbf{3 x}-\mathbf{4 y} \leq \mathbf{1 2}$.
2. Shade the solution set of $\boldsymbol{x}+\boldsymbol{y}>\mathbf{1}$.
3. Shade the solution set of $\left\{\begin{array}{ll}\boldsymbol{y} & \geq-\mathbf{1} \\ \boldsymbol{y} & \leq \mathbf{3}\end{array}\right.$ and

## STOP!

1. Shade the solution set of $\mathbf{3 x}-4 \boldsymbol{y} \leq 12$.

Solution:


| $x$ | $y$ | $3 x-4 y=12$ | the boundary |
| :--- | :--- | :--- | :--- |
| $\mathbf{0}$ | $-\mathbf{3}$ | $\mathbf{3 ( 0 )}-\mathbf{4 ( - 3 ) = 1 2}$ | $y$ intercept. |
| $\mathbf{4}$ | $\mathbf{0}$ | $\mathbf{3 ( 4 ) - 4 ( 0 ) = 1 2}$ | $x$ intercept. |

Easiest point to test: $(\mathbf{0}, \mathbf{0})$ (if not on boundary.) $3(0)-4(0) \leq 12$ or $0 \leq 12$ is true.
Shade the region containing $(\mathbf{0}, \mathbf{0})$.
2. Shade the solution set of $\boldsymbol{x}+\boldsymbol{y}>\mathbf{1}$.

Solution:


| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{x}+\boldsymbol{y}=\mathbf{1}$ | the boundary |
| :--- | :--- | :--- | :--- |
| $\mathbf{0}$ | $\mathbf{1}$ | $(\mathbf{0})+(\mathbf{1})=\mathbf{1}$ | the $-\boldsymbol{y}$ intercept. |
| $\mathbf{1}$ | $\mathbf{0}$ | $(\mathbf{1})+(\mathbf{0})=\mathbf{1}$ | the $-\boldsymbol{x}$ intercept. |
| $\mathbf{4}$ | $\mathbf{- 3}$ | $(\mathbf{4})+(\mathbf{- 3})=\mathbf{1}$ |  |

Use $(\mathbf{0}, \mathbf{0})$ as test point (if not on the boundary.)
$\boldsymbol{x}+\boldsymbol{y}>\mathbf{1}$ becomes $\mathbf{0}+\mathbf{0}>\mathbf{1}$ which is false,
so shade the region not containing the test point.
Points on the boundary don't satisfy this inequality.
$=$ is not part of $\boldsymbol{x}+\boldsymbol{y}>\mathbf{1}$. Draw the boundary dashed.

## Example 3:

Shade the solution set of $\left\{\begin{array}{ll}y & \geq-1\end{array}\right.$ and


## Chapter 57

## Direct and Inverse Variation

### 57.1 Youtube

https://www.youtube.com/playlist?list=PL7EB5650F21254AFE\&feature=view_all

### 57.2 Basics

### 57.2.1 Direct Variation

The standard legal speed limit in my state is $\mathbf{6 5} \mathrm{mph}$
(the is Rate (speed) $\boldsymbol{R}=\frac{\boldsymbol{D} \text { miles }}{\boldsymbol{T} \text { hours }}$ ) or $\boldsymbol{R T}=\boldsymbol{D}$.
Suppose $\boldsymbol{R}$ is kept constant at $\mathbf{6 5} \mathrm{mph}$, then the relation between $\boldsymbol{D}$ and $\boldsymbol{T}$ is said to illustrate direct variation. $\boldsymbol{D}$ is also said to be directly proportional to $\boldsymbol{T}$.

$$
\boldsymbol{D} \sim \boldsymbol{T} \text { and in general } \boldsymbol{T}=\boldsymbol{k} \boldsymbol{D} \text { where } \boldsymbol{k} \text { is the constant of propotionality. }
$$

In $\boldsymbol{T}=\mathbf{1}$ hour the distance covered is $\boldsymbol{D}=\mathbf{6 5}$ miles.
In $\boldsymbol{T}=\mathbf{2}$ hours the distance covered is $\boldsymbol{D}=\mathbf{2 ( 6 5 )}$ miles.
In $\boldsymbol{T}=\mathbf{3}$ hours the distance covered is $\boldsymbol{D}=\mathbf{3}(\mathbf{6 5 )}$ miles.
If $\boldsymbol{T}$ is multiplied by a number, then $\boldsymbol{D}$ is multiplied by the same number.
Note: $65=\frac{65}{1}=\frac{2(65)}{2}=\frac{3(65)}{3}=\frac{D_{1}}{T_{1}}=\frac{D_{2}}{T_{2}}$.
Do you see "proportions"?

### 57.2.2 Inverse Variation

Let's come back to the standard legal speed limit in my state which is 65 mph (the is Rate (speed) $\boldsymbol{R}=\frac{\boldsymbol{D} \text { miles }}{\boldsymbol{T} \text { hours }}$ ).

Suppose $\boldsymbol{D}$ is kept constant at 130 miles, then the relation between $\boldsymbol{R}$ and $\boldsymbol{T}$ is said to illustrate indirect variation. $\boldsymbol{R}$ is also said to be inversely proportional to $\boldsymbol{T}$.
$\boldsymbol{R} \sim \frac{\mathbf{1}}{\boldsymbol{T}}$ and in general $\boldsymbol{R}=\frac{\boldsymbol{k}}{\boldsymbol{T}}$ where $\boldsymbol{k}$ is the constant of propotionality.
At $\boldsymbol{R}=\mathbf{3 2 . 5} \mathrm{mph}$ it takes $\boldsymbol{T}=\mathbf{4}$ hours to travel 130 miles.
At $\boldsymbol{R}=\mathbf{2 ( 3 2 . 5 )}=\mathbf{6 5} \mathrm{mph}$ it takes $\boldsymbol{T}=\frac{\mathbf{4}}{\mathbf{2}}=\mathbf{2}$ hours to travel 130 miles.
At $\boldsymbol{R}=\mathbf{4 ( 3 2 . 5 )}=\mathbf{1 3 0} \mathrm{mph}$ it takes $\boldsymbol{T}=\frac{\mathbf{4}}{\mathbf{4}}=\mathbf{1}$ hour to travel 130 miles.
If $\boldsymbol{R}$ is multiplied by a number, then $\boldsymbol{D}$ is divided by the same number.
Note: $130=(32.5)(4)=(65)(2)=(130)(1)=R_{1} T_{1}=R_{2} T_{2}$.

### 57.2.3 Joint Variation

If $\boldsymbol{z} \sim \boldsymbol{x}$ and $\boldsymbol{z} \sim \frac{\mathbf{1}}{\boldsymbol{y}}$ then $\boldsymbol{z}=\frac{\boldsymbol{k} \boldsymbol{x}}{\boldsymbol{y}}$.
In words, if $\boldsymbol{z}$ is directly proportional to $\boldsymbol{x}$ and (at the same time) $\boldsymbol{z}$ is inversely proportional to $\boldsymbol{y}$ then $\boldsymbol{z}$ is jointly proportional to $\boldsymbol{x}$ and $\frac{\mathbf{1}}{\boldsymbol{y}}$.

As an example $\boldsymbol{F}=\frac{\boldsymbol{G} \boldsymbol{m}_{1} \boldsymbol{m}_{\mathbf{2}}}{\boldsymbol{R}^{2}}$ where $\boldsymbol{G}$ is the constant of variation, $\boldsymbol{F}$ is the force of attraction between two bodies of mass $\boldsymbol{m}_{\mathbf{1}}$ and $\boldsymbol{m}_{\mathbf{2}}$ located a distance $\boldsymbol{R}$ apart. Thus if either $\boldsymbol{m}_{\boldsymbol{1}}$ and/or $\boldsymbol{m}_{\mathbf{2}}$ increase, then $\boldsymbol{F}$ increases. If the distance between the bodies increases, then the force decreases (as the square of the distance).

### 57.3 Examples

Example 1:
Your wages $\boldsymbol{W}$ are directly proportional to the number of hours $\boldsymbol{N}$ you worked. $\boldsymbol{W} \sim \boldsymbol{N}$.
You work $\boldsymbol{N}=\mathbf{1 5}$ hours and are paid $\boldsymbol{W}=\$ \mathbf{1 5 0}$. How much money will you earn if you work $\mathbf{3 7 . 5}$ hours?

## Solution:

$$
W=k N
$$

. We are given $\mathbf{1 5 0}=\boldsymbol{k}(\mathbf{1 5 )}$ from which $\boldsymbol{k}=\mathbf{1 0}$.
Our formula now reads

$$
W=10 N
$$

If $\boldsymbol{N}=\mathbf{3 7 . 5}$ then $\boldsymbol{W}=\mathbf{1 0 ( 3 7 . 5 )}=\mathbf{\$ 3 7 5}$ (before deductions).
Example 2:
The distance $\boldsymbol{D}$ an object falls in free fall (discounting air resistance) is proportional (that is directly proportional) to the square of the time $\boldsymbol{T}$ in motion. $\boldsymbol{D} \sim \boldsymbol{T}^{\mathbf{2}}$.

The Foresthill Bridge, the tallest in California, is about 730 ft above the American River. You drop a stone from the bridge. The stone falls 16 ft in $\mathbf{1}$ second. How long will it take for the stone to reach the water?

## Solution:

$$
D=k T^{2}
$$

We are given $\mathbf{1 6}=\boldsymbol{k}(\mathbf{1})$ from which $\boldsymbol{k}=\mathbf{1 6}$.
Our formula now reads

$$
D=16 T^{2}
$$

If $D=730$ then $730=16 T^{2}$ or $T^{2}=\frac{\mathbf{7 3 0}}{16}=45.6$, thus $T \approx 7$ seconds.
Example 3:
The Intensity $\boldsymbol{I}$ of a light source varies inversely as the square of the distance $\boldsymbol{r}$ from a light source.
If the intensity is $\mathbf{1 2}$ lumens at a distance of $\mathbf{5} \mathrm{m}$, what is the intensity when the distance is $\mathbf{1 0} \mathrm{m}$ ? Solution:

$$
I=\frac{k}{r^{2}}
$$

We are given $\mathbf{1 2}=\frac{\boldsymbol{k}}{\mathbf{5}^{2}}$ from which $\boldsymbol{k}=(\mathbf{1 2})(\mathbf{2 5})=\mathbf{3 0 0}$.
Our formula now reads

$$
I=\frac{300}{r^{2}}
$$

If $\boldsymbol{r}=\mathbf{1 0}$ then $\boldsymbol{I}=\frac{\mathbf{3 0 0}}{\mathbf{1 0}^{\mathbf{2}}}$ or $\boldsymbol{I}=\mathbf{3}$ lumens.
Example 4:
The force of attraction $\boldsymbol{F}$ between two bodies of masses $\boldsymbol{m}_{\mathbf{1}}$ and $\boldsymbol{m}_{\mathbf{2}}$ is directly proportional to those masses and inversely proportional to the distance between these bodies $\boldsymbol{r}^{\mathbf{2}}$.

$$
F=\frac{G m_{1} m_{2}}{r^{2}}
$$

$\boldsymbol{F}=$ gravitational force between the earth and the moon,
$\boldsymbol{G}=$ Universal gravitational constant (of proportionality) $=6.67 \times 10^{-11} \frac{\mathrm{Nm}^{2}}{(\mathrm{~kg})^{2}}$,
$\boldsymbol{m}_{1}=$ mass of the moon $=7.36 \times \mathbf{1 0}^{\mathbf{2 2}} \mathrm{kg}$,
$\boldsymbol{m}_{\mathbf{2}}=$ mass of the earth $=\mathbf{5 . 9 7 4 2} \times \mathbf{1 0}^{\mathbf{2 4}}$ and
$r=$ distance between the earth and the moon $=\mathbf{3 8 4}, 402 \mathrm{~km}$
$F=\frac{\left(6.67 \times 10^{-11}\right)\left(7.36 \times 10^{22}\right)\left(5.9742 \times 10^{24}\right)}{(384,402)^{2}}=1.985 \times 10^{26} \mathrm{~N}$
Read more:
http://wiki.answers.com/Q/What_is_the_gravitational_force
_between_the_Earth_and_the_moon\#ixzz1RvqVpeEh

### 57.4 Exercise 57

1. Your wages $\boldsymbol{W}$ are directly proportional to the number of hours $\boldsymbol{N}$ you worked. $\boldsymbol{W} \sim \boldsymbol{N}$.

You worked $\boldsymbol{N}=\mathbf{2 5}$ hours and were paid $\boldsymbol{W}=\$ \mathbf{3 0 0}$. How much money will you earn if you work 27.5 hours?
2. The distance $\boldsymbol{D}$ an object falls in free fall (discounting air resistance) is proportional (that is directly proportional) to the square of the time $\boldsymbol{T}$ in motion. $\boldsymbol{D} \sim \boldsymbol{T}^{\mathbf{2}}$.
The Golden Gate Bridge in San Francisco tower has a height of $\mathbf{7 4 6} \mathrm{ft}$ above the water. You drop a stone from the bridge. The stone falls 16 ft in the first $\mathbf{1}$ second. How long will it take for the stone to reach the water?
3. The Intensity $\boldsymbol{I}$ of a light source varies inversely as the square of the distance $\boldsymbol{r}$ from a light source. If the intensity is $\mathbf{1 8}$ lumens at a distance of $\mathbf{3} \mathrm{m}$, what is the intensity when the distance is $\mathbf{6} \mathrm{m}$ ?
4. The force of attraction $\boldsymbol{F}$ between two bodies of masses $\boldsymbol{m}_{\boldsymbol{1}}$ and $\boldsymbol{m}_{\boldsymbol{2}}$ is directly proportional to those masses and inversely proportional to the distance between these bodies $\boldsymbol{r}^{2}$.
How will $\boldsymbol{F}$ change if $\boldsymbol{m}_{\mathbf{1}}$ is doubled, $\boldsymbol{m}_{\mathbf{2}}$ is tripled, and the distance is divided by $\mathbf{4}$.

## STOP!

1. Your wages $\boldsymbol{W}$ are directly proportional to the number of hours $\boldsymbol{N}$ you worked. $\boldsymbol{W} \sim \boldsymbol{N}$.

You worked $\boldsymbol{N}=\mathbf{2 5}$ hours and were paid $\boldsymbol{W}=\mathbf{\$ 3 0 0}$. How much money will you earn if you work 27.5 hours?

## Solution:

$$
W=k N
$$

We are given $\mathbf{3 0 0}=\boldsymbol{k}(\mathbf{2 5})$ from which $\boldsymbol{k}=\mathbf{1 2}$.
Our formula now reads

$$
w=12 N
$$

If $\boldsymbol{N}=\mathbf{2 7 . 5}$ then $\boldsymbol{W}=\mathbf{1 2 ( 2 7 . 5 )}=\$ 330$ (before deductions).
2. The distance $\boldsymbol{D}$ an object falls in free fall (discounting air resistance) is proportional (that is directly proportional) to the square of the time $\boldsymbol{T}$ in motion. $\boldsymbol{D} \sim \boldsymbol{T}^{\mathbf{2}}$.
The Golden Gate Bridge in San Francisco tower has a height of $\mathbf{7 4 6} \mathrm{ft}$ above the water. You drop a stone from the bridge. The stone falls $\mathbf{1 6} \mathrm{ft}$ in the first $\mathbf{1}$ second. How long will it take for the stone to reach the water?
Solution:

$$
D=k T^{2}
$$

We are given $\mathbf{1 6}=\boldsymbol{k}(\mathbf{1})$ from which $\boldsymbol{k}=\mathbf{1 6}$.
Our formula now reads

$$
D=16 T^{2}
$$

If $D=746$ then $746=16 T^{2}$ or $T^{2}=\frac{\mathbf{7 4 6}}{16}=46.6$, thus $T \approx 7$ seconds.
3. The Intensity $\boldsymbol{I}$ of a light source varies inversely as the square of the distance $\boldsymbol{r}$ from a light source.

If the intensity is $\mathbf{1 8}$ lumens at a distance of $\mathbf{3} \mathrm{m}$, what is the intensity when the distance is $\mathbf{6} \mathrm{m}$ ?
Solution:

$$
I=\frac{k}{r^{2}}
$$

We are given $\mathbf{1 8}=\frac{\boldsymbol{k}}{\mathbf{3}^{2}}$ from which $\boldsymbol{k}=(\mathbf{1 8})(\mathbf{9})=162$.
Our formula now reads

$$
I=\frac{162}{r^{2}}
$$

If $r=6$ then $I=\frac{\mathbf{1 6 2}}{6^{2}}$ or $I=4.5$ lumens.
4. The force of attraction $\boldsymbol{F}$ between two bodies of masses $\boldsymbol{m}_{\mathbf{1}}$ and $\boldsymbol{m}_{\mathbf{2}}$ is directly proportional to those masses and inversely proportional to the distance between these bodies $\boldsymbol{r}^{2}$.
How will $\boldsymbol{F}$ change if $\boldsymbol{m}_{\mathbf{1}}$ is doubled, $\boldsymbol{m}_{\boldsymbol{2}}$ is tripled, and the distance is divided by 4 .

## Solution:

$$
\begin{aligned}
& M_{1}=2 m_{1} \\
& M_{2}=3 m_{1} \\
& R=0.25 r \\
& F_{\text {new }}=\frac{G M_{1} M_{2}}{R^{2}} \\
& =\frac{(G)\left(2 m_{1}\right)\left(3 m_{2}\right)}{(0.25)^{2} r^{2}} \\
& =\frac{(G)\left(2 m_{1}\right)\left(3 m_{2}\right)}{(0.25)^{2} r^{2}} \\
& =\frac{6 G m_{1} m_{2}}{0.0625 r^{2}} \\
& = \\
& \\
& =96 F_{\mathrm{old}}
\end{aligned}
$$

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## Chapter 58

## Introduction to (Square, cubic, ...) Roots and Radical Expressions

### 58.1 Youtube

https://www.youtube.com/playlist?list=PLC48453B7313FD2D7\&feature=view_all

### 58.2 Basics

The square root of $\mathbf{6 4}$ is $\mathbf{8}$ because $\mathbf{8}^{\mathbf{2}}=\mathbf{6 4}$. The square root of $\mathbf{6 4}$ is NOT $-\mathbf{8}$, in spite of what some authors maintain and in spite of $(-8)^{2}=64$. If $x^{2}=64$, then $x= \pm \sqrt{64}= \pm 8$. The negative sign enters here because the variable can be $\sqrt{\mathbf{6 4}}=\mathbf{8}$ or its opposite $-\sqrt{\mathbf{6 4}}=-\mathbf{8}$.
$\sqrt{\mathbf{4 9}}=\sqrt{\mathbf{7}^{2}}=7$. Note that the square factors $\mathbf{7}$ come out of the square root as a single $\mathbf{7}$.
Similarly for higher roots:
$\sqrt[3]{8}=\sqrt[3]{\mathbf{2}^{3}}=2$ A cube factor comes out of the cube root as a single factor.
$\sqrt[4]{81}=\sqrt[4]{3^{4}}=4 \mathrm{~A}$ fourth power factor comes out of the fourth root as a single factor.
Even roots are never negative.
Odd roots can be positive or negative. $\sqrt[5]{\mathbf{3 2}}=\sqrt[5]{\mathbf{2}^{5}}=\mathbf{2}$ and $\sqrt[5]{-\mathbf{3 2}}=\sqrt[5]{(-2)^{5}}=\mathbf{- 2}$.
We'll come back to higher order roots later in another chapter. From here on we discus square roots.
$\sqrt{25}=\sqrt{5^{2}}=5$
$-\sqrt{25}=-\sqrt{5^{2}}=-5$
$\sqrt{-25}$ is not a real number because $5^{2} \neq \mathbf{2 5}$ and neither is $(-5)^{\mathbf{2}}$.
$\sqrt{\mathbf{3 6}}=\sqrt{6^{2}}=6$ and note that
$\sqrt{36}=\sqrt{4 \cdot 9}=\sqrt{2^{2} \cdot 3^{2}}=2 \cdot 3=6$ also.
This last example illustrates

$$
\sqrt{\mathbf{a}^{2} \mathbf{b}^{2}}=\sqrt{\mathbf{a}^{2}} \sqrt{\mathbf{b}^{2}}=\mathbf{a b} \text { (assuming } \mathbf{a}>0 \text { and } \mathbf{b}>0 . \text {.) }
$$

but


This is worth repeating:
$\sqrt{a b}=\sqrt{a} \sqrt{b}$ but $\sqrt{a \pm b} \neq \sqrt{a} \pm \sqrt{b}$

$$
\sqrt{\frac{a^{2}}{b^{2}}}=\frac{\sqrt{a^{2}}}{\sqrt{b^{2}}}=\frac{a}{b}(\text { assuming } a \geq 0 \text { and } b \geq 0
$$

$(\sqrt{5})^{2}=(\sqrt{5})(\sqrt{5})=\sqrt{5^{2}}=5$
$(\sqrt{7})^{2}=(\sqrt{7})(\sqrt{7})=\sqrt{7^{2}}=7$
$(\sqrt{x})^{2}=(\sqrt{x})(\sqrt{x})=\sqrt{x^{2}}=x$ (assuming $\left.x \geq 0\right)$.

$$
(\sqrt{x})^{2}=\sqrt{x^{2}}=x \geq 0
$$

What is $\sqrt{7}$ ? According to a calculator $\sqrt{7}=2.645751311 \cdots$ where $\cdots$ refers to infinitely many digits no group of which repeats.

What is $(2.645751311 \cdots)^{2}$ ?
A word on (my) calculator oddity:
$\sqrt{\mathbf{1 3 3}}=11.53256259$ as shown in the display. Now I clear my calculator and enter $\mathbf{1 1 . 5 3 2 5 6 2 5 9}$.
Squaring this entry leads to $\mathbf{1 3 2 . 9 9 9 9 9 9 9 9 9}$ instead of $\mathbf{1 3}$. This is due to $\sqrt{\mathbf{1 3 3}}$ having infinitely many
digits. My calculator works with one or two digits more than displayed. Rounding usually gives the desired result ( $\mathbf{1 3}$ here), but not always.

My calculator gives the exact answer of $\mathbf{1 3}$ when I square $\sqrt{\mathbf{1 3}}$ without clearing the display before squaring.

### 58.3 Examples

Example 1:
Without using the square button on a calculator, find $\sqrt{\mathbf{2 1 3}, \mathbf{4 4 4}}$ if you are told that $\mathbf{2 1 3}, \mathbf{4 4 4}$ is a perfect square.

## Solution:

Find the prime factorization of $\mathbf{2 1 3}, \mathbf{4 4 4}$ and extract pairs of factors as single factors.

| $\mathbf{2 1 3}, \mathbf{4 4 4}$ | $\mathbf{2}$ | divide by 2 to get 106,722 |
| ---: | :--- | :--- |
| $\mathbf{1 0 6}, \mathbf{7 2 2}$ | $\mathbf{2}$ | divide by 2 to get 53,361 |
| $\mathbf{5 3 , 3 6 1}$ | $\mathbf{3}$ | divide by 3 to get 17,787 |
| $\mathbf{1 7 , \mathbf { 7 8 7 }}$ | $\mathbf{3}$ | divide by 3 to get 5,929 |
| $\mathbf{5 , 9 2 9}$ | $\mathbf{7}$ | divide by 7 to get 847 |
| $\mathbf{8 4 7}$ | $\mathbf{7}$ | divide by 11 to get 121 |
| $\mathbf{1 2 1}$ | $\mathbf{1 1}$ | divide by 11 to get 11 |
| $\mathbf{1 1}$ | $\mathbf{1 1}$ | divide by 11 to get 1 |
| $\mathbf{1}$ |  |  |

Thus $\sqrt{213,444}=\sqrt{2^{2} \cdot 3^{2} \cdot 7^{2} \cdot 11^{2}}=2 \cdot 3 \cdot 7 \cdot 11=462$.
Check: $\mathbf{4 6 2}^{\mathbf{2}}=\mathbf{2 1 3}, 444$.

Example 2:
Simplify $7 \sqrt{2^{3} a^{2} b^{5}}$

## Solution:

$$
\begin{aligned}
7 \sqrt{2^{3} a^{2} b^{5}} & =7 \sqrt{(2)^{2}(2)\left(a^{2}\right)\left(b^{2}\right)^{2}(b)} \\
& =7 \cdot(2)(a)\left(b^{2}\right) \sqrt{(2)(b)} \\
& =14 a b^{2} \sqrt{2 b}
\end{aligned}
$$

Example 3:
Simplify $\sqrt{98(x+3)^{2}}$

## Solution:

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$$
\begin{aligned}
\sqrt{98(x+3)^{2}} & =\sqrt{2 \cdot 7^{2}(x+3)^{2}} \\
& =7(x+3) \sqrt{2} \quad \text { assuming } \mathrm{x} \geq-3
\end{aligned}
$$

Example 4:
Simplify $-5 \sqrt{45}$

## Solution:

$$
\begin{aligned}
-5 \sqrt{45} & =-5 \sqrt{3^{2} \cdot 5} \\
& =-5 \cdot 3 \sqrt{5} \\
& =-15 \sqrt{5}
\end{aligned}
$$

Example 5:
Simplify $a^{2} \sqrt{a^{2}+2 a b+b^{2}}$.

## Solution:

$$
\begin{aligned}
a^{2} \sqrt{a^{2}+2 a b+b^{2}} & =a^{2} \sqrt{(a+b)^{2}} \\
& =a^{2}|a+b| \text { absolute value guarantees } \\
& \text { that } a+b \geq \mathbf{0}
\end{aligned}
$$

Example 6:
Simplify $\sqrt{\mathbf{0 . 0 9}}$

## Solution:

$$
\sqrt{0.09}=\sqrt{(0.3)^{2}}=0.3
$$

### 58.4 Exercise 57

1. Without using the square root button on a calculator, find $\sqrt{\mathbf{1 0}, \mathbf{7 3 2}, \mathbf{1 7 6}}$ if you are told that $\mathbf{1 0}, \mathbf{7 3 2}, \mathbf{1 7 6}$ is a perfect square.
2. Simplify $9 \sqrt{3^{5} a^{3} b^{4}}$
3. Simplify $\sqrt{242(y-5)^{2}}$
4. Simplify $-7(\sqrt{\mathbf{8 0}})^{2}$
5. Simplify $x^{3} \sqrt{\boldsymbol{x}^{2}-\mathbf{2 x y}+\boldsymbol{y}^{2}}$.
6. Simplify $\sqrt{0.0004}$

## STOP!

1. Without using the square root button on a calculator, find $\sqrt{\mathbf{1 0}, \mathbf{7 3 2}, \mathbf{1 7 6}}$ if you are told that $10,732,176$ is a perfect square.

## Solution:

Find the prime factorization of $\mathbf{1 0}, \mathbf{7 3 2}, \mathbf{1 7 6}$ and extract square factors as single factors.

| $\mathbf{1 0}, \mathbf{7 3 2}, \mathbf{1 7 6}$ | $\mathbf{2}$ | divide by 2 |
| ---: | :--- | :--- |
| $\mathbf{5}, \mathbf{3 6 6}, \mathbf{0 8 8}$ | $\mathbf{2}$ | divide by 2 |
| $\mathbf{2 , 6 8 3}, \mathbf{0 4 4}$ | $\mathbf{2}$ | divide by 2 |
| $\mathbf{1}, \mathbf{3 4 1}, \mathbf{5 2 2}$ | $\mathbf{2}$ | divide by 2 |
| $\mathbf{6 7 0 , 7 6 1}$ | $\mathbf{3}$ | divide by 3 |
| $\mathbf{2 2 3}, \mathbf{5 8 7}$ | $\mathbf{3}$ | divide by 3 |
| $\mathbf{7 4 , 5 2 9}$ | $\mathbf{7}$ | divide by 7 |
| $\mathbf{1 0 , 6 4 7}$ | $\mathbf{7}$ | divide by 7 |
| $\mathbf{1 , 5 2 1}$ | $\mathbf{1 3}$ | divide by 13 |
| $\mathbf{1 1 7}$ | $\mathbf{3}$ | divide by 3 |
| $\mathbf{3 9}$ | $\mathbf{3}$ | divide by 3 |
| $\mathbf{1 3}$ | $\mathbf{1 3}$ | divide by 13 |
| $\mathbf{1}$ |  |  |

Thus
$\sqrt{10,732,176}=\sqrt{2^{4} \cdot 3^{4} \cdot 7^{2} \cdot 13^{2}}=2^{2} \cdot 3^{2} \cdot 7 \cdot 13=3,276$.
Check: $\mathbf{3 ,} \mathbf{2 7 6}^{\mathbf{2}}=\mathbf{1 0}, \mathbf{7 3 2}, 176$.
2. Simplify $9 \sqrt{3^{5} a^{3} b^{4}}$

Solution:

$$
\begin{aligned}
9 \sqrt{3^{5} a^{3} b^{4}} & =9 \sqrt{\left(3^{2}\right)^{2}(3)\left(a^{2}\right)(a)\left(b^{2}\right)^{2}} \\
& =9 \cdot 3^{2} a b^{2} \sqrt{(3)(a)} \\
& =81 a b^{2} \sqrt{3 a}
\end{aligned}
$$

3. Simplify $\sqrt{242(y-5)^{2}}$

Solution:

$$
\begin{aligned}
\sqrt{242(y-5)^{2}} & =\sqrt{2 \cdot 11^{2}(y-5)^{2}} \\
& =11|y-5| \sqrt{2}
\end{aligned}
$$

4. Simplify $-7(\sqrt{\mathbf{8 0}})^{2}$

Solution:

$$
-7(\sqrt{80})^{2}=-7(80)=-560
$$

5. Simplify $x^{3} \sqrt{x^{2}-2 x y+y^{2}}$.

Solution:

$$
\begin{aligned}
x^{3} \sqrt{x^{2}-2 x y+y^{2}} & =x^{3} \sqrt{(x-y)^{2}} \\
& =x^{3}(x-y) \text { if } x-y \geq 0
\end{aligned}
$$

6. Simplify $\sqrt{\mathbf{0 . 0 0 0 4}}$

Solution:

$$
\sqrt{0.0004}=\sqrt{(0.02)^{2}}=0.02
$$

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## Chapter 59

## Simplification of Radical Expressions

### 59.1 Youtube

https://www.youtube.com/playlist?list=PLF12A785F8DAC1F69\&feature=view_all

### 59.2 Basics

As pointed out before, the square of a factor (a pair of identical factors) comes out of the square root as a single factor $\sqrt{3^{2} a^{3} b}=3 a \sqrt{a b}$. Remember that factor means multiplication.
$\sqrt{a^{2} \pm b^{2}}=a \pm b$ is incorrect.

### 59.3 Examples

Example 1:
Simplify $\sqrt{\mathbf{2 8 8} \boldsymbol{x}^{\mathbf{5}} \boldsymbol{y}^{\mathbf{2}} \boldsymbol{z}}$. Assume $\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{z} \geq \mathbf{0}$.

## Solution:

$$
\begin{aligned}
\sqrt{288 x^{5} y^{2} z} & =\sqrt{2 \cdot 12^{2}\left(x^{2}\right)^{2} x y^{2} z} \\
& =12 x^{2} y \sqrt{2 x z}
\end{aligned}
$$

Example 2:

Assume $\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}, \boldsymbol{z} \geq \mathbf{0}$

Method 1:

$$
\begin{aligned}
\sqrt{\frac{27 x^{-3} y^{2}}{3 x^{3} y^{2}}} & =\sqrt{\frac{\left(3^{2}\right)(3) y^{2}}{3 x^{3} x^{3} y^{2}}} \\
& =\sqrt{\frac{(3)^{2}}{\left(x^{3}\right)^{2}}} \\
& =\frac{\sqrt{(3)^{2}}}{\sqrt{\left(x^{3}\right)^{2}}} \\
& =\frac{3}{x^{3}}
\end{aligned}
$$

Method 2:

$$
\begin{aligned}
\sqrt{\frac{27 x^{-3} y^{2}}{3 x^{3} y^{2}}} & =\frac{\sqrt{\left(3^{2}\right) x^{-3}(3) y^{2}}}{\sqrt{3 x^{3} y^{2}}} \\
& =\frac{\sqrt{\left(3^{2}\right) x^{-4} x^{1}(3) y^{2}}}{\sqrt{3\left(x^{2}\right) x y^{2}}} \\
& =\frac{3 x^{-2} y \sqrt{x(3)}}{x y \sqrt{3 x}} \\
& =\frac{3 x^{-2}}{x} \\
& =\frac{3}{x x^{2}}=\frac{3}{x^{3}}
\end{aligned}
$$

Example 3:
Simplify $\sqrt{\left(0.1 x^{3}\right)\left(0.016 x^{5}\right)(x+5)^{2}}$. Assume $\boldsymbol{x} \geq 0$.

## Solution:

$$
\begin{aligned}
\sqrt{\left(0.1 x^{3}\right)\left(0.016 x^{5}\right)(x+5)^{2}} & =\sqrt{(0.1)(0.016)\left(x^{5} x^{3}\right)(x+5)^{2}} \\
& =\sqrt{(0.0016)\left(x^{8}\right)(x+5)^{2}} \\
& =\sqrt{(0.04)^{2}\left(x^{4}\right)^{2}(x+5)^{2}} \\
& =(0.04)\left(x^{4}\right)|x+5|
\end{aligned}
$$

### 59.4 Exercise 59

1. Simplify $\sqrt{\mathbf{5 4 a ^ { 7 }} \boldsymbol{b}^{4} \boldsymbol{c}^{\mathbf{3}}}$. Assume $\boldsymbol{a} \geq \mathbf{0}, \boldsymbol{b} \geq \mathbf{0}, \boldsymbol{c} \geq \mathbf{0}$.
2. Simplify $\sqrt{\frac{\mathbf{3 2 x ^ { - 7 } \boldsymbol { y } ^ { \mathbf { 3 } }}}{\mathbf{2} \boldsymbol{x}^{3} \boldsymbol{y}^{\mathbf{5}}}}$. Leave no negative exponents.

Assume $\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}$.
3. Simplify $\sqrt{\left(0.5 a^{3}\right)\left(0.125 a^{5}\right)(a-7)^{2}}$

## STOP!

1. Simplify $\sqrt{\mathbf{5 4 \boldsymbol { a } ^ { 7 }} \boldsymbol{b}^{\mathbf{4}} \boldsymbol{c}^{\mathbf{3}}}$. Assume $\boldsymbol{a} \geq \mathbf{0}, \boldsymbol{b} \geq \mathbf{0}, \boldsymbol{c} \geq \mathbf{0}$.

## Solution:

$$
\begin{aligned}
\sqrt{54 a^{7} b^{4} c^{3}} & =\sqrt{9 \cdot 6 a^{6} a b^{4} c^{2} c} \\
& =3 a^{3} b^{2} c \sqrt{6 a c}
\end{aligned}
$$


Assume $\boldsymbol{x} \geq \mathbf{0}, \boldsymbol{y} \geq \mathbf{0}$.

$$
\begin{aligned}
\sqrt{\frac{32 x^{-7} y^{3}}{2 x^{3} y^{5}}} & =\sqrt{\frac{2 \cdot 16 x^{-7}}{2 x^{3} y^{5-3}}} \\
& =\sqrt{\frac{16}{x^{3} x^{7} y^{2}}} \\
& =\sqrt{\frac{4^{2}}{x^{10} y^{2}}} \\
& =\frac{4}{x^{5} y}
\end{aligned}
$$

3. Simplify $\sqrt{\left(0.5 a^{3}\right)\left(0.125 a^{5}\right)(a-7)^{2}}$

Solution:

$$
\begin{aligned}
\sqrt{\left(0.5 a^{3}\right)\left(0.125 a^{5}\right)(a-7)^{2}} & =\sqrt{(0.5)(0.125)\left(a^{3}\right)\left(a^{5}\right)(a-7)^{2}} \\
& =\sqrt{(0.0625)\left(a^{8}\right)(a-7)^{2}} \\
& =0.25 a^{4}|a-7|
\end{aligned}
$$

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## Chapter 60

# Multiplication/Division of Radical Expressions 

60.1 Youtube
https://www.youtube.com/playlist?list=PL5C37A9A693D96317\&feature=view_all
60.2 Basics

Recall that

$$
\begin{gathered}
\sqrt{a^{2}}=|a| \\
\sqrt{a b}=\sqrt{a} \cdot \sqrt{b} \\
\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}
\end{gathered}
$$

### 60.3 Examples

Example 1:
Find the product $\sqrt{\mathbf{3 5}} \sqrt{\mathbf{5 6}} \sqrt{\mathbf{4 0}}$
Solution:

$$
\begin{aligned}
\sqrt{35} \sqrt{56} \sqrt{40} & =\sqrt{5 \cdot 7} \sqrt{7 \cdot 2 \cdot 2^{2}} \sqrt{2^{2} \cdot 2 \cdot 5} \\
& =\sqrt{5 \cdot 7 \cdot 7 \cdot 2 \cdot 2^{2} \cdot 2^{2} \cdot 2 \cdot 5} \\
& =\sqrt{5^{2} \cdot 7^{2} \cdot 2^{2} \cdot 2^{2} \cdot 2^{2}} \\
& =2^{3} \cdot 5 \cdot 7=280
\end{aligned}
$$

## Example 2:

Simplify $(5 \sqrt{15})(7 \sqrt{55})$

## Solution:

$$
\begin{aligned}
(5 \sqrt{15})(7 \sqrt{55}) & =(5 \sqrt{3 \cdot 5})(7 \sqrt{5 \cdot 11}) \\
& =5 \cdot 7 \sqrt{3 \cdot 5 \cdot 5 \cdot 11} \\
& =35 \cdot 5 \sqrt{3 \cdot 11} \\
& =175 \sqrt{33}
\end{aligned}
$$

Example 3:
Simplify $\frac{\sqrt{6 x^{3} y}}{\sqrt{5 z^{4}}} \div \frac{\sqrt{18 x^{5}}}{\sqrt{20 y^{3}}}$
Solution:

$$
\begin{aligned}
\frac{\sqrt{6 x^{3} y}}{\sqrt{5 z^{4}}} \div \frac{\sqrt{18 x^{5}}}{\sqrt{20 y^{3}}} & =\frac{\sqrt{6 x^{3} y}}{\sqrt{5 z^{4}}} \cdot \frac{\sqrt{20 y^{3}}}{\sqrt{18 x^{5}}} \\
& =\sqrt{\frac{2 \cdot 3 \cdot 2^{2} \cdot 5 x^{3} y^{4}}{3^{2} \cdot 2 \cdot 5 z^{4} x^{5}}} \\
& =\sqrt{\frac{3 \cdot 2^{2} y^{4}}{3^{2} z^{4} x^{2}}} \\
& =\frac{2 y^{2} \sqrt{3}}{3 x z^{2}}
\end{aligned}
$$

### 60.4 Exercise 60

1. Find the product $\sqrt{\mathbf{4 5}} \sqrt{\mathbf{3 0}} \sqrt{\mathbf{9 8}}$
2. Simplify $(3 \sqrt{65})(7 \sqrt{26})(2 \sqrt{2})$
3. Simplify $\frac{\sqrt{4 a^{5} b^{3}}}{\sqrt{6 c^{6}}} \div \frac{\sqrt{32 a^{4}}}{\sqrt{3 b^{7} c^{3}}}$

Assume $\boldsymbol{a}>\mathbf{0}, \boldsymbol{b}>\mathbf{0}, \boldsymbol{c}>\mathbf{0}$.

1. Find the product $\sqrt{\mathbf{4 5}} \sqrt{\mathbf{3 0}} \sqrt{\mathbf{9 8}}$

## Solution:

$$
\begin{aligned}
\sqrt{45} \sqrt{30} \sqrt{98} & =\sqrt{3^{2} \cdot 5} \sqrt{2 \cdot 3 \cdot 5} \sqrt{2 \cdot 7^{2}} \\
& =\sqrt{2^{2} \cdot 3^{2} \cdot 5^{2} \cdot 7^{2}} \sqrt{3} \\
& =2 \cdot 3 \cdot 5 \cdot 7 \sqrt{3} \\
& =210 \sqrt{3}
\end{aligned}
$$

2. Simplify $(3 \sqrt{65})(7 \sqrt{26})(2 \sqrt{2})$

## Solution:

$$
\begin{aligned}
(3 \sqrt{65})(7 \sqrt{26})(2 \sqrt{2}) & =(3 \sqrt{5 \cdot 13})(7 \sqrt{2 \cdot 13})(2 \sqrt{2}) \\
& =2 \cdot 3 \cdot 7 \sqrt{5 \cdot 13} \sqrt{2 \cdot 13} \sqrt{2} \\
& =42 \sqrt{5 \cdot 13^{2} \cdot 2^{2}} \\
& =2 \cdot 13 \cdot 42 \sqrt{5} \\
& =1,092 \sqrt{5}
\end{aligned}
$$

3. Simplify $\frac{\sqrt{4 a^{5} b^{3}}}{\sqrt{6 c^{6}}} \div \frac{\sqrt{32 a^{4}}}{\sqrt{3 b^{7} c^{3}}}$. Assume $a>0, b>0, c>0$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt{4 a^{5} b^{3}}}{\sqrt{6 c^{6}}} \div \frac{\sqrt{32 a^{4}}}{\sqrt{3 b^{7} c^{3}}} & =\frac{\sqrt{4 a^{5} b^{3}}}{\sqrt{6 c^{6}}} \cdot \frac{\sqrt{3 b^{7} c^{3}}}{\sqrt{32 a^{4}}} \\
& =\sqrt{\frac{4 a^{5} b^{3}}{6 c^{6}} \cdot \frac{3 b^{7} c^{3}}{32 a^{4}}} \\
& =\sqrt{\frac{a^{5} b^{3}}{2 c^{6}} \cdot \frac{b^{7} c^{3}}{8 a^{4}}} \\
& =\sqrt{\frac{a^{5-4} b^{3+7}}{16 c^{6-3}}} \\
& =\sqrt{\frac{a b^{10}}{4^{2} c^{3}}} \\
& =\frac{b^{5}}{4 c} \sqrt{\frac{a}{c}}
\end{aligned}
$$

## Chapter 61

## Rationalizing the Denominator

### 61.1 Youtube

https://www.youtube.com/playlist?list=PL30B574FA54E481B1\&feature=view_all

### 61.2 Basics

Let's add fractions with square roots in the denominator. You'll see why the next development is cumbersome and will be avoided.

Add $\frac{2}{\sqrt{3}}+\frac{5}{\sqrt{7}}+\frac{4}{11}$
We know we need the same denominator:

$$
\frac{2}{\sqrt{3}}+\frac{5}{11 \sqrt{7}}+\frac{4}{11}=\frac{2}{\sqrt{3}} \cdot \frac{11 \sqrt{7}}{11 \sqrt{7}}+\frac{5}{\sqrt{7}} \cdot \frac{11 \sqrt{3}}{11 \sqrt{3}}+\frac{4}{11} \cdot \frac{\sqrt{3} \sqrt{7}}{\sqrt{3} \sqrt{7}}
$$

There is no need to saddle the last fraction's denominator with square roots if we rationalize the denominators.
Rationalizing denominators is a process by which roots are eliminated from denominators.

$$
\begin{aligned}
& \frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}} \cdot 1=\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}=\frac{2 \sqrt{3}}{(\sqrt{3})^{2}}=\frac{2 \sqrt{3}}{3} \\
& \frac{5}{\sqrt{7}}=\frac{5}{\sqrt{7}} \cdot 1=\frac{5}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}=\frac{5 \sqrt{7}}{(\sqrt{7})^{2}}=\frac{5 \sqrt{7}}{7}
\end{aligned}
$$

thus

$$
\begin{aligned}
\frac{2}{\sqrt{3}}+\frac{5}{\sqrt{7}}+\frac{4}{11} & =\frac{2 \sqrt{3}}{3}+\frac{5 \sqrt{7}}{7}+\frac{4}{11} \\
& =\frac{2 \sqrt{3}}{3} \cdot \frac{7 \cdot 11}{7 \cdot 11}+\frac{5 \sqrt{7}}{7} \cdot \frac{3 \cdot 11}{3 \cdot 11}+\frac{4}{11} \cdot \frac{3 \cdot 7}{3 \cdot 7}
\end{aligned}
$$

This last form looks a lot cleaner than the first. It has only two square roots as compared with ten square roots in the earlier expression.

### 61.3 Examples

Example 1:
Rationalize $\frac{12}{\sqrt{18}}$

## Solution:

$$
\begin{aligned}
\frac{12}{\sqrt{18}} & =\frac{12}{\sqrt{18}} \frac{\sqrt{18}}{\sqrt{18}} \\
& =\frac{12 \sqrt{18}}{(\sqrt{18})^{2}} \\
& =\frac{12 \sqrt{18}}{(\sqrt{18})^{2}} \\
& =\frac{12 \sqrt{18}}{18} \\
& =\frac{2 \sqrt{2 \cdot 3^{2}}}{3}=\frac{2 \cdot 3 \sqrt{2}}{3}=2 \sqrt{2}
\end{aligned}
$$

Wow! I wish we had replaced $\sqrt{\mathbf{1 8}}$ by $\mathbf{3} \sqrt{\mathbf{2}}$ immediately at the beginning.

$$
\begin{aligned}
\frac{12}{\sqrt{18}} & =\frac{12}{\sqrt{2 \cdot 3^{2}}} \\
& =\frac{12}{3 \sqrt{2}} \\
& =\frac{4}{\sqrt{2}} \\
& =\frac{4}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} \\
& =\frac{4 \sqrt{2}}{2}=2 \sqrt{2}
\end{aligned}
$$

Example 2:

Rationalize $\frac{35 \sqrt{x}}{\sqrt{7 y^{3}}}$

## Solution:

$$
\begin{aligned}
\frac{35 \sqrt{x}}{\sqrt{7 y^{3}}} & =\frac{35 \sqrt{x}}{y \sqrt{7 y}} \\
& =\frac{35 \sqrt{x}}{y \sqrt{7 y}} \cdot \frac{\sqrt{7 y}}{\sqrt{7 y}} \\
& =\frac{35 \sqrt{7 x y}}{y(\sqrt{7 y})^{2}} \\
& =\frac{35 \sqrt{7 x y}}{7 y^{2}} \\
& =\frac{5 \sqrt{7 x y}}{y^{2}}
\end{aligned}
$$

Example 3:
Rationalize $\frac{10}{5-\sqrt{3}}$

## Solution:

Why is the following method not recommended?

$$
\begin{aligned}
\frac{10}{5-\sqrt{3}} & =\frac{10}{5-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
& =\frac{10 \sqrt{3}}{5 \sqrt{3}-(\sqrt{3})^{2}} \\
& =\frac{10 \sqrt{3}}{5 \sqrt{3}-3} \text { We still have a square root in the denominator. }
\end{aligned}
$$

Let's try another approach (which will still not lead to success).

$$
\begin{aligned}
\frac{10}{5-\sqrt{3}} & =\frac{10}{5-\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} \\
& =\frac{10 \cdot 5-10 \sqrt{3}}{(5-\sqrt{3})^{2}} \\
& =\frac{10 \cdot 5-10 \sqrt{3}}{25-10 \sqrt{3}+(\sqrt{3})^{2}} \\
& =\frac{10 \cdot 5-10 \sqrt{3}}{28-10 \sqrt{3}} \text { can you "cancel" } 10 \sqrt{3} ? \text { Definitely not! }
\end{aligned}
$$

We get rid of the square root in the denominator if we multiply numerator and denominator by the conjugate of the denominator.
$\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{a}-\boldsymbol{b}$ are conjugates.
Note that $(a+b)(a-b)=a^{2}-b^{2}$
$5-\sqrt{3}$ and $5+\sqrt{3}$ are conjugates.
Note that $(5-\sqrt{3})(5+\sqrt{3})=5^{2}-(\sqrt{3})^{2}=25-3=22$ (no square roots).

$$
\begin{aligned}
\frac{10}{5-\sqrt{3}} & =\frac{10}{5-\sqrt{3}} \cdot \frac{5+\sqrt{3}}{5+\sqrt{3}} \\
& =\frac{10 \cdot 5+10 \sqrt{3}}{5^{2}-(\sqrt{3})^{2}} \\
& =\frac{10 \cdot 5+10 \sqrt{3}}{25-3} \\
& =\frac{50+10 \sqrt{3}}{22} \\
& =\frac{10(5+\sqrt{3})}{2 \cdot 11} \\
& =\frac{2 \cdot 5(5+\sqrt{3})}{2 \cdot 11} \\
& =\frac{2}{2} \cdot \frac{5(5+\sqrt{3})}{11} \\
& =\frac{1 \cdot \frac{5(5+\sqrt{3})}{11}}{} \\
= & \frac{5(5+\sqrt{3})}{11}
\end{aligned}
$$

### 61.4 Exercise 61

1. Rationalize $\frac{\mathbf{2 0}}{\sqrt{\mathbf{2 8}}}$
2. Rationalize $\frac{63 \sqrt{a}}{\sqrt{3 a b^{5}}}$
3. Rationalize $\frac{8}{\sqrt{7}+\sqrt{5}}$

## STOP!

1. Rationalize $\frac{\mathbf{2 0}}{\sqrt{28}}$

$$
\begin{aligned}
& \text { Solution: } \\
& \begin{aligned}
\frac{20}{\sqrt{28}} & =\frac{2^{2} \cdot 5}{\sqrt{2^{2} \cdot 7}} \\
& =\frac{2^{2} \cdot 5}{2 \sqrt{\cdot 7}} \\
& =\frac{2 \cdot 5}{\sqrt{7}} \\
& =\frac{2 \cdot 5}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} \\
& =\frac{10 \sqrt{7}}{7}
\end{aligned}
\end{aligned}
$$

2. Rationalize $\frac{63 \sqrt{a}}{\sqrt{3 a b^{5}}}$

$$
\begin{aligned}
& \text { Solution: } \\
& \begin{aligned}
\frac{63 \sqrt{a}}{\sqrt{3 a b^{5}}} & =63 \sqrt{\frac{a}{3 a b^{5}}} \\
& =\frac{63}{b^{2}} \cdot \sqrt{\frac{1}{3 b}} \\
& =\frac{63}{b^{2}} \cdot \sqrt{\frac{1}{3 b}} \cdot \frac{\sqrt{3 b}}{\sqrt{3 b}} \\
& =\frac{63}{b^{2}} \cdot \frac{\sqrt{3 b}}{3 b} \\
& =\frac{21}{b^{2}} \cdot \frac{\sqrt{3 b}}{b} \\
& =\frac{21 \sqrt{3 b}}{b^{3}}
\end{aligned}
\end{aligned}
$$

3. Rationalize $\frac{8}{\sqrt{7}+\sqrt{5}}$

Solution:

$$
\begin{aligned}
\frac{8}{\sqrt{7}+\sqrt{5}} & =\frac{8}{\sqrt{7}+\sqrt{5}} \cdot \frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}} \\
& =\frac{8 \sqrt{7}-8 \sqrt{5}}{(\sqrt{7})^{2}-(\sqrt{5})^{2}} \\
& =\frac{8 \sqrt{7}-8 \sqrt{5}}{7-5} \\
& =\frac{8(\sqrt{7}-\sqrt{5})}{2} \\
& =4(\sqrt{7}-\sqrt{5})
\end{aligned}
$$

## Chapter 62

## Adding and Subtracting Radicals

### 62.1 Youtube

https://www.youtube.com/playlist?list=PLB4D8CC40165228EA\&feature=view_all

### 62.2 Basics

Simplify each radical as much as possible before forming a sum or difference.
$\sqrt{\mathbf{9} \boldsymbol{x}^{\mathbf{3}}}$ simplifies to $\mathbf{3 x} \sqrt{\boldsymbol{x}}$ We need not mention the restriction $\boldsymbol{x} \geq \mathbf{0}$. $\boldsymbol{x}$ is understood not to negative otherwise neither square root would be defined.

### 62.3 Examples

Example 1:
Simplify $4 \sqrt{x^{5}}+3 x \sqrt{x^{3}}-5 x^{2} \sqrt{x}$

## Solution:

$$
\begin{aligned}
4 \sqrt{x^{5}}+3 x \sqrt{x^{3}}-5 x^{2} \sqrt{x} & =4 x^{2} \sqrt{x}+3 x^{2} \sqrt{x}-5 x^{2} \sqrt{x} \\
& =(4+3-5) x^{2} \sqrt{x} \\
& =2 x^{2} \sqrt{x}
\end{aligned}
$$

Example 2:
Simplify $\sqrt{\mathbf{7 5}}-\sqrt{\mathbf{2 7}}+\sqrt{\mathbf{1 0 8}}$

## Solution:

$$
\begin{aligned}
\sqrt{75}-\sqrt{27}+\sqrt{108} & =\sqrt{3 \cdot 5^{2}}-\sqrt{3 \cdot 3^{2}}+\sqrt{3 \cdot 6^{2}} \\
& =5 \sqrt{3}-3 \sqrt{3}+6 \sqrt{3} \\
& =(5-3+6) \sqrt{3}=8 \sqrt{3}
\end{aligned}
$$

Example 3:
Simplify $\sqrt{\frac{25 x^{3} y^{2}}{4}}-x \sqrt{\frac{16 x y^{2}}{9}}+x y \sqrt{\frac{49 x}{36}}$
Solution:

$$
\begin{aligned}
& \sqrt{\frac{25 x^{3} y^{2}}{4}}-x \sqrt{\frac{16 x y^{2}}{9}}+x y \sqrt{\frac{49 x}{36}} \\
= & \frac{5 x y}{2} \sqrt{x}-\frac{4 x y}{3} \sqrt{x}+\frac{7 x y}{6} \sqrt{x} \\
= & \frac{6 \cdot 5 x y}{6 \cdot 2} \sqrt{x}-\frac{4 \cdot 4 x y}{4 \cdot 3} \sqrt{x}+\frac{2 \cdot 7 x y}{2 \cdot 6} \sqrt{x} \\
= & \frac{30 x y}{12} \sqrt{x}-\frac{16 x y}{12} \sqrt{x}+\frac{14 x y}{12} \sqrt{x} \\
= & \frac{(30-16+14) x y \sqrt{x}}{12} \\
= & \frac{28 x y \sqrt{x}}{12}=\frac{7 x y \sqrt{x}}{3}
\end{aligned}
$$

Example 4:
Simplify $\sqrt{0.01}+\sqrt{0.09}+\sqrt{0.25}-\sqrt{0.64}$

## Solution:

$$
\sqrt{0.01}+\sqrt{0.09}+\sqrt{0.25}-\sqrt{0.64}=0.1+0.3+0.5-0.8=0.1
$$

Example 5:
Simplify $a \sqrt{a^{3} b^{5}}+a^{2} b \sqrt{a b^{3}}-2 a^{2} b^{2} \sqrt{a b}$

## Solution:

$$
\begin{aligned}
& a \sqrt{a^{3} b^{5}}+a^{2} b \sqrt{a b^{3}}-2 a^{2} b^{2} \sqrt{a b} \\
= & a a b^{2} \sqrt{a b}+a^{2} b b \sqrt{a b}-2 a^{2} b^{2} \sqrt{a b} \\
= & a^{2} b^{2} \sqrt{a b}+a^{2} b^{2} \sqrt{a b}-2 a^{2} b^{2} \sqrt{a b} \\
= & (1+1-2) a^{2} b^{2} \sqrt{a b}=0
\end{aligned}
$$

Example 6:
Simplify $x^{2} y^{2} \sqrt{\left(x^{2}+y^{2}\right)^{2}}-x^{4} \sqrt{4 y^{4}}+y^{4} \sqrt{9 x^{4}}-x^{2} y^{2} \sqrt{\left(4 y^{2}-x^{2}\right)^{2}}$

## Solution:

$$
\begin{aligned}
& x^{2} y^{2} \sqrt{\left(x^{2}+y^{2}\right)^{2}}-x^{4} \sqrt{4 y^{4}}+y^{4} \sqrt{9 x^{4}}-x^{2} y^{2} \sqrt{\left(4 y^{2}-x^{2}\right)^{2}} \\
= & x^{2} y^{2}\left(x^{2}+y^{2}\right)-x^{4}\left(2 y^{2}\right)+y^{4}\left(3 x^{2}\right)-x^{2} y^{2}\left(4 y^{2}-x^{2}\right) \\
= & x^{4} y^{2}+x^{2} y^{4}-2 x^{4} y^{2}+3 x^{2} y^{4}-4 x^{2} y^{4}+x^{4} y^{2} \\
= & \underline{x^{4} y^{2}}+\underline{\underline{x^{2} y^{4}}}-\underline{2 x^{4} y^{2}}+\underline{\underline{3 x^{2} y^{4}}}-\underline{\underline{4 x^{2} y^{4}}}+\underline{x^{4} y^{2}} \\
= & 0
\end{aligned}
$$

### 62.4 Exercise 62

1. Simplify $7 \sqrt{a^{7}}+5 a \sqrt{a^{5}}-3 a^{3} \sqrt{a}$
2. Simplify $\sqrt{\mathbf{6 3}}-\sqrt{\mathbf{1 7 5}}+\sqrt{\mathbf{8 4 7}}$
3. Simplify $\sqrt{\frac{36 a^{30} b^{20}}{64}}-a \sqrt{\frac{9 a^{10} b^{20}}{81}}+a b \sqrt{\frac{121 a^{10}}{11}}$
4. Simplify $\sqrt{0.0025}+\sqrt{0.0081}-\sqrt{0.36}-\sqrt{0.04}$
5. Simplify $u^{2} \sqrt{u^{6} v^{7}}+u^{3} v^{2} \sqrt{u^{4} v^{3}}-2 u v^{3} \sqrt{u^{8} v}$
6. Simplify

$$
\left(x^{2}-y^{2}\right) \sqrt{\left(x^{2}+y^{2}\right)^{2}}-x^{4} \sqrt{9 y^{4}}+y^{4} \sqrt{25 x^{4}}-x^{2} y^{2} \sqrt{\left(5 y^{2}-3 x^{2}\right)^{2}}
$$

## STOP!

1. Simplify $7 \sqrt{a^{7}}+5 a \sqrt{a^{5}}-3 a^{3} \sqrt{a}$

Solution:

$$
\begin{aligned}
7 \sqrt{a^{7}}+5 a \sqrt{a^{5}}-3 a^{3} \sqrt{a} & =7 a^{3} \sqrt{a}+5 a a^{2} \sqrt{a}-3 a^{3} \sqrt{a} \\
& =(7+5-3) a^{3} \sqrt{a}=9 a^{3} \sqrt{a}
\end{aligned}
$$

2. Simplify $\sqrt{63}-\sqrt{175}+\sqrt{847}$

Solution:

$$
\begin{aligned}
\sqrt{63}-\sqrt{175}+\sqrt{847} & =\sqrt{3^{2} \cdot 7}-\sqrt{5^{2} \cdot 7}+\sqrt{11^{2} \cdot 7} \\
& =3 \sqrt{7}-5 \sqrt{7}+11 \sqrt{7} \\
& =(3-5+11) \sqrt{7}=9 \sqrt{7}
\end{aligned}
$$

3. Simplify $\sqrt{\frac{36 a^{30} b^{20}}{64}}-a \sqrt{\frac{9 a^{10} b^{20}}{81}}+a b \sqrt{\frac{121 a^{10}}{11}}$

Solution:

$$
\begin{aligned}
& \sqrt{\frac{36 a^{30} b^{20}}{64}}-a \sqrt{\frac{9 a^{10} b^{20}}{81}}+a b \sqrt{\frac{121 a^{10}}{11}} \\
= & \sqrt{\frac{9 a^{30} b^{20}}{16}}-a \sqrt{\frac{a^{10} b^{20}}{9}}+a b \sqrt{\frac{11 a^{10}}{1}} \\
= & \frac{3 a^{15} b^{10}}{4}-\frac{a a^{5} b^{10}}{3}+a b \sqrt{11} a^{5} \\
= & \frac{3 a^{15} b^{10}}{4}-\frac{a^{6} b^{10}}{3}+\sqrt{11} a^{6} b
\end{aligned}
$$

4. Simplify $\sqrt{\mathbf{0 . 0 0 2 5}}+\sqrt{0.0081}-\sqrt{0.36}-\sqrt{0.04}$

## Solution:

$$
\begin{aligned}
& \sqrt{0.0025}+\sqrt{0.0081}-\sqrt{0.36}-\sqrt{0.04} \\
= & 0.05+0.09-0.6-0.2 \\
= & 0.14-0.8=-0.66
\end{aligned}
$$

5. Simplify $u^{2} \sqrt{u^{6} v^{7}}+u^{3} v^{2} \sqrt{u^{4} v^{3}}-2 u v^{3} \sqrt{u^{8} v}$

Solution:

$$
\begin{aligned}
& u^{2} \sqrt{u^{6} v^{7}}+u^{3} v^{2} \sqrt{u^{4} v^{3}}-2 u v^{3} \sqrt{u^{8} v} \\
= & u^{2} u^{3} \sqrt{v^{6} v}+u^{3} u^{2} v^{2} \sqrt{v^{2} v}-2 u u^{4} v^{3} \sqrt{v} \\
= & u^{5} v^{3} \sqrt{v}+u^{5} v^{2} v \sqrt{v}-2 u^{5} v^{3} \sqrt{v} \\
= & u^{5} v^{3} \sqrt{v}+u^{5} v^{3} \sqrt{v}-2 u^{5} v^{3} \sqrt{v}=0
\end{aligned}
$$

6. Simplify

$$
\left(x^{2}-y^{2}\right) \sqrt{\left(x^{2}+y^{2}\right)^{2}}-x^{4} \sqrt{9 y^{4}}+y^{4} \sqrt{25 x^{4}}-x^{2} y^{2} \sqrt{\left(5 y^{2}-3 x^{2}\right)^{2}}
$$

## Solution:

$$
\begin{aligned}
& \left(x^{2}-y^{2}\right) \sqrt{\left(x^{2}+y^{2}\right)^{2}}-x^{4} \sqrt{9 y^{4}}+y^{4} \sqrt{25 x^{4}}-x^{2} y^{2} \sqrt{\left(5 y^{2}-3 x^{2}\right)^{2}} \\
& =\left(x^{2}-y^{2}\right)\left(x^{2}+y^{2}\right)-3 x^{4} y^{2}+5 x^{2} y^{4}-x^{2} y^{2}\left(5 y^{2}-3 x^{2}\right) \\
& =x^{4}-y^{4}-\underline{3 x^{4} y^{2}}+\underline{\underline{5 x^{2} y^{4}}}-\underline{\underline{5 x^{2} y^{4}}}+\underline{3 x^{4} y^{2}} \\
& =x^{4}-y^{4}
\end{aligned}
$$

## Chapter 63

## Radical Expressions with Several Terms

### 63.1 Youtube

https://www.youtube.com/playlist?list=PL3A97F7B2F57DB217\&feature=view_all

### 63.2 Basics

Recall that

$$
\begin{gathered}
\sqrt{a^{2}}=|a| \\
\hline \sqrt{a b}=\sqrt{a} \cdot \sqrt{b} \\
\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}
\end{gathered}
$$

### 63.3 Examples

Example 1:
Simplify $7 \sqrt{11}-5 \sqrt{11}+\sqrt{11}$

## Solution:

How would you solve $\mathbf{7 x}-5 \boldsymbol{x}+\boldsymbol{x}$ ?
The point is that we deal with like-radicals just like we deal with like-term.

$$
7 \sqrt{11}-5 \sqrt{11}+\sqrt{11}=(7-5+1) \sqrt{11}=3 \sqrt{11}
$$

Example 2:
Simplify $5 x \sqrt{x}-7 \sqrt{x^{3}}$

## Solution:

$$
\begin{aligned}
5 x \sqrt{x}-7 \sqrt{x^{3}} & =5 x \sqrt{x}-7 x \sqrt{x} \\
& =-2 x \sqrt{x}
\end{aligned}
$$

Why do we not need to use absolute value or mention that
$\boldsymbol{x} \geq \mathbf{0}$ in the statement of the problem?
Answer: If $\boldsymbol{x}<\mathbf{0}$ then the original problem statement with $\sqrt{\boldsymbol{x}}$ is not real.
Example 3:
Simplify $\sqrt{3 x}\left(3 \sqrt{75 x^{3}}-7 x \sqrt{12 x}\right)$

## Solution:

$$
\begin{aligned}
& \sqrt{3 x}\left(3 \sqrt{75 x^{3}}-7 x \sqrt{12 x}\right) \\
= & 3 \sqrt{(3 x)\left(75 x^{3}\right)}-7 x \sqrt{(3 x)(12 x)} \\
= & 3 \sqrt{(3 x)\left(3 \cdot 5^{2} x^{2} x\right)}-7 x \sqrt{(3 x)\left(3 \cdot 2^{2} x\right)} \\
= & 3 \cdot 3 \cdot 5 x(x)-2(7)(3) x(x) \\
= & 45 x^{2}-42 x^{2}=3 x^{2}
\end{aligned}
$$

Example 4:
Simplify $(2 \sqrt{5 x+1}+3 \sqrt{7 x+1})(2 \sqrt{5 x+1}-3 \sqrt{7 x+1})$

## Solution:

$$
\begin{aligned}
& (2 \sqrt{5 x+1}+3 \sqrt{7 x+1})(2 \sqrt{5 x+1}-3 \sqrt{7 x+1}) \\
= & (2 \sqrt{5 x+1}+3 \sqrt{7 x+1})(2 \sqrt{5 x+1}-3 \sqrt{7 x+1}) \\
= & (2 \sqrt{5 x+1})^{2}-(3 \sqrt{7 x+1})^{2} \\
= & 4(5 x+1)-9(7 x+1) \\
= & 20 x+4-63 x-9 \\
= & -43 x-5
\end{aligned}
$$

Example 5:
Rationalize the denominator $\frac{\sqrt{6}}{\sqrt{5 a^{6}}}$
Assume $\boldsymbol{a}>\mathbf{0}$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt{6}}{\sqrt{5 a^{6}}} & =\frac{\sqrt{6}}{a^{3} \sqrt{5}} \\
& =\frac{\sqrt{6}}{a^{3} \sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\
& =\frac{\sqrt{30}}{5 a^{3}}
\end{aligned}
$$

Example 6:
Rationalize the denominator $\frac{\sqrt{6}}{\sqrt{5}+\sqrt{3 x}}$
Assume $\boldsymbol{a}>\mathbf{0}$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt{6}}{\sqrt{5}+\sqrt{3 x}} & =\frac{\sqrt{6}}{\sqrt{5}+\sqrt{3 x}} \cdot \frac{\sqrt{5}-\sqrt{3 x}}{\sqrt{5}-\sqrt{3 x}} \\
& =\frac{\sqrt{6}(\sqrt{5}-\sqrt{3 x})}{(\sqrt{5}+\sqrt{3 x})(\sqrt{5}-\sqrt{3 x})} \\
& =\frac{\sqrt{30}-\sqrt{(3 \cdot 2)(3 x)}}{(\sqrt{5})^{2}-(\sqrt{3 x})^{2}} \\
& =\frac{\sqrt{30}-3 \sqrt{2 x}}{5-3 x}
\end{aligned}
$$

### 63.4 Exercise 63

1. Simplify $4 \sqrt{3}-6 \sqrt{3}+-\sqrt{3}$
2. Simplify $9 a \sqrt{a^{3}}-2 \sqrt{a^{5}}$
3. Simplify $(7 \sqrt{5 a}+4 \sqrt{2 b})\left(3 \sqrt{2 a^{3}}-9 a \sqrt{b}\right)$
4. Simplify $(3 \sqrt{2 a-1}+5 \sqrt{2 a+1})(3 \sqrt{2 a-1}-5 \sqrt{2 a+1})$
5. Rationalize the denominator $\frac{\sqrt{\mathbf{1 3}}}{\sqrt{\mathbf{3 9 a}^{\mathbf{3} \boldsymbol{b}}}}$
6. Rationalize the denominator $\frac{\sqrt{12 x}}{\sqrt{5 x}-3 \sqrt{2 x}}$

## STOP!

1. Simplify $4 \sqrt{3}-6 \sqrt{3}+-\sqrt{3}$

## Solution:

How would you solve $\mathbf{4 x}-\boldsymbol{6 x}-\boldsymbol{x}$ ?
The point is that we deal with like-radicals just like we deal with like-term.

$$
\begin{aligned}
4 \sqrt{3}-6 \sqrt{3}+-\sqrt{3} & =(4-6-1) \sqrt{3} \\
& =-3 \sqrt{3}
\end{aligned}
$$

2. Simplify $9 a \sqrt{a^{3}}-2 \sqrt{a^{5}}$

Solution:

$$
\begin{aligned}
9 a \sqrt{a^{3}}-2 \sqrt{a^{5}} & =9 a \sqrt{a^{2} a}-2 \sqrt{a^{4} a} \\
& =9 a^{2} \sqrt{a}-2 a^{2} \sqrt{a} \\
& =7 a^{2} \sqrt{a}
\end{aligned}
$$

3. Simplify $(7 \sqrt{5 a}+4 \sqrt{2 b})\left(3 \sqrt{2 a^{3}}-9 a \sqrt{b}\right)$

Solution:

$$
\begin{aligned}
& (7 \sqrt{5 a}+4 \sqrt{2 b})\left(3 \sqrt{2 a^{3}}-9 a \sqrt{b}\right) \\
= & 7 \sqrt{5 a}\left(3 \sqrt{2 a^{3}}-9 a \sqrt{b}\right)+4 \sqrt{2 b}\left(3 \sqrt{2 a^{3}}-9 a \sqrt{b}\right) \\
= & (7 \sqrt{5 a})\left(3 \sqrt{2 a^{3}}\right)-(7 \sqrt{5 a})(9 a \sqrt{b})+(4 \sqrt{2 b})\left(3 \sqrt{2 a^{3}}\right) \\
& -(4 \sqrt{2 b})(9 a \sqrt{b}) \\
= & 21 \sqrt{5 a \cdot 2 a^{3}}-63 a \sqrt{5 a b}+12 \sqrt{2 b \cdot 2 a^{3}}-36 a \sqrt{2 b b} \\
= & 21 \sqrt{10 a^{4}}-63 a \sqrt{5 a b}+12 \cdot 2 \sqrt{b a^{3}}-36 a \sqrt{2 b^{2}} \\
= & 21 a^{2} \sqrt{10}-63 a \sqrt{5 a b}+24 a \sqrt{b a}-36 a b \sqrt{2}
\end{aligned}
$$

There are no radical like-terms.
4. Simplify $(3 \sqrt{2 a-1}+5 \sqrt{2 a+1})(3 \sqrt{2 a-1}-5 \sqrt{2 a+1})$

## Solution:

$$
\begin{aligned}
& (3 \sqrt{2 a-1}+5 \sqrt{2 a+1})(3 \sqrt{2 a-1}-5 \sqrt{2 a+1}) \\
= & (3 \sqrt{2 a-1})^{2}-(5 \sqrt{2 a+1})^{2} \\
= & 9(2 a-1)-25(2 a+1) \\
= & 18 a-9-50 a-25 \\
= & -32 a-34
\end{aligned}
$$

5. Rationalize the denominator $\frac{\sqrt{\mathbf{1 3}}}{\sqrt{\mathbf{3 9 a ^ { 3 }} \boldsymbol{b}}}$

Assume $\boldsymbol{a}>\mathbf{0}$ and $\boldsymbol{b}>\mathbf{0}$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt{13}}{\sqrt{39 a^{3} b}} & =\sqrt{\frac{13}{39 a^{2} a b}} \\
& =\frac{1}{a} \sqrt{\frac{1}{3 a b}} \\
& =\frac{1}{a} \sqrt{\frac{3 a b}{(3 a b)(3 a b)}} \\
& =\frac{\sqrt{3 a b}}{a(3 a b)} \\
& =\frac{\sqrt{3 a b}}{3 a^{2} b}
\end{aligned}
$$

6. Rationalize the denominator $\frac{\sqrt{12 x}}{\sqrt{5 x}-3 \sqrt{2 x}}$

Assume $\boldsymbol{a}>\mathbf{0}$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt{12 x}}{\sqrt{5 x}-3 \sqrt{2 x}} & =\frac{\sqrt{2^{2} \cdot 3 x}}{\sqrt{5 x}-3 \sqrt{2 x}} \\
& =\frac{2 \sqrt{3 x}}{\sqrt{5 x}-3 \sqrt{2 x}} \cdot \frac{\sqrt{5 x}+3 \sqrt{2 x}}{\sqrt{5 x}+3 \sqrt{2 x}} \\
& =\frac{2 \sqrt{3 x}(\sqrt{5 x}+3 \sqrt{2 x})}{(\sqrt{5 x}-3 \sqrt{2 x})(\sqrt{5 x}+3 \sqrt{2 x})} \\
& =\frac{2 \sqrt{15 x^{2}}+6 \sqrt{6 x^{2}}}{(\sqrt{5 x})^{2}-(3 \sqrt{2 x})^{2}} \\
& =\frac{2 x \sqrt{15}+6 x \sqrt{6}}{5 x-9(2 x)} \\
& =\frac{2 x(\sqrt{15}+3 \sqrt{6})}{5 x-18 x} \\
& =\frac{2 x(\sqrt{15}+3 \sqrt{6})}{-13 x} \\
& =-\frac{2}{13}(\sqrt{15}+3 \sqrt{6})
\end{aligned}
$$

## Chapter 64

## Solving Radical Equations

### 64.1 Youtube

https://www.youtube.com/playlist?list=PL42F72B2CBEB7973A\&feature=view_all

### 64.2 Basics

Solve $\sqrt{\boldsymbol{x}}=\mathbf{5}$.
Answer: $\boldsymbol{x}=\mathbf{2 5}$. This quick guess was quite simple. But we need to develop a technique to solve more challenging situations.

Given

$$
\begin{aligned}
\sqrt{\boldsymbol{x}} & =\mathbf{5} & & \text { Multiply both sides by } \mathbf{5 .} \\
\mathbf{5} \sqrt{\boldsymbol{x}} & =(5)^{2} & & \text { Replace } \mathbf{5} \text { on the left by } \sqrt{\boldsymbol{x}} \text { which is }=\mathbf{5} . \\
(\sqrt{\boldsymbol{x}})^{2} & =(\mathbf{5})^{2} & & \text { Replace } \mathbf{5} \text { on the left by } \sqrt{\boldsymbol{x}} \text { which is }=\mathbf{5} . \\
\boldsymbol{x} & =\mathbf{2 5} & & \text { We are not finished. }
\end{aligned}
$$

Note that we went from $\sqrt{\boldsymbol{x}}=5$ to $(\sqrt{\boldsymbol{x}})^{2}=(5)^{2}$ which means we squared both sides.
It is imperative that we check the answer(s). The process of squaring both sides may have introduced extraneous solutions.

Suppose we are given $\sqrt{\boldsymbol{x}}=\mathbf{- 1}$. Then squaring both sides leads to $\boldsymbol{x}=\mathbf{1}$. Checking $\boldsymbol{x}=\mathbf{1}$ leads to $\sqrt{\mathbf{1}}=\mathbf{- 1}$ which is false. Thus $\boldsymbol{x}=\mathbf{1}$ is extraneous.

### 64.3 Examples

Example 1:
Solve $\mathbf{3} \sqrt{x+5}-8=0$.

## Solution:

$$
\begin{aligned}
& \mathbf{3} \sqrt{\boldsymbol{x}+5}-\mathbf{8}=\mathbf{0} \quad \text { Isolate the square root term. } \\
& 3 \sqrt{x+5}=8 \quad \text { Square both sides. } \\
& 9(\sqrt{x+5})^{2}=8^{2} \\
& 9(x+5)=64 \\
& x+5=\frac{64}{9} \\
& x=\frac{64}{9}-\frac{5}{1} \\
& x=\frac{64}{9}-\frac{45}{9} \\
& =\frac{\mathbf{6 4 - 4 5}}{9}=\frac{19}{9} \quad \text { We must check this answer. } \\
& 3 \sqrt{x+5}-8=0 \\
& 3 \sqrt{\frac{19}{9}+5}-8=0 \\
& 3 \sqrt{\frac{19+45}{9}}-8=0 \\
& 3 \sqrt{\frac{64}{9}}-8=0 \\
& 3\left(\frac{8}{3}\right)-8=0 \\
& 8-8=0
\end{aligned}
$$

So $x=8$ is a (the) solution.
If the original problem had been
"Solve $\mathbf{3} \sqrt{\boldsymbol{x}+5}+\mathbf{8}=\mathbf{0}$ " then $\mathbf{3} \sqrt{\boldsymbol{x}+5}=-\mathbf{8}$ would have extraneous solution(s). $\mathbf{3} \sqrt{\boldsymbol{x}+\mathbf{5}}=-\mathbf{8}$ is not possible (in the real number system) because the square root is negative.

Example 2:
Solve $\sqrt{5 x}+8=13$.

## Solution:

$$
\begin{aligned}
& \sqrt{5 x}+8=13 \text { Isolate square root term. Subtract } 8 \text { from both sides. } \\
& \sqrt{5 x}=5 \quad \text { Square both sides. } \\
& 5 x=25 \\
& \text { Divide by } 5 . \\
& x=5 \quad \text { Divide by } 5 .
\end{aligned}
$$

Now check 5:

$$
\begin{aligned}
\sqrt{5 x}+8 & =13 \\
\sqrt{5 \cdot 5}+8 & =13 \\
5+8 & =13
\end{aligned}
$$

So $\boldsymbol{x}=\mathbf{5}$ is a (the) solution.
Example 3:
Solve $\sqrt{x+7}-\sqrt{x+14}+1=0$.

## Solution:

Method 1 (not recommended):

$$
\begin{aligned}
\sqrt{x+7}-\sqrt{x+14}+1 & =0 \text { Subtract } 1 . \\
\sqrt{x+7}-\sqrt{x+14} & =-1 \text { Square both sides. } \\
(\sqrt{x+7}-\sqrt{x+14})^{2} & =(-1)^{2} \\
(\sqrt{x+7})^{2}-2(\sqrt{x+7})(\sqrt{x+14})+(\sqrt{x+14})^{2} & =1 \\
x+7-2 \sqrt{(x+7)(x+14)}+x+14 & =1 \\
2 x+21-2 \sqrt{(x+7)(x+14)} & =1 \\
-2 \sqrt{(x+7)(x+14)} & =-2 x-20 \text { Factor } 2 . \text { Then divide by }-2 . \\
\sqrt{(x+7)(x+14)} & =x+10 \text { Square again. } \\
(x+7)(x+14) & =x^{2}+20 x+100 \\
x^{2}+21 x+198 & =x^{2}+20 x+100 \\
21 x+98+ & =20 x+100 \\
x & =2
\end{aligned}
$$

Now check:

$$
\begin{array}{r}
\sqrt{2+7}-\sqrt{2+14}+1=0 \\
\sqrt{9}-\sqrt{16}+1=0 \\
3-4+1=0
\end{array}
$$

Thus $\boldsymbol{x}=\mathbf{2}$ is a (the) solution.
Method 2:
Isolate a square root,

$$
\begin{array}{rlll}
\sqrt{x+7}-\sqrt{x+14}+1 & =0 & \\
\sqrt{x+7} & =-1+\sqrt{x+14} & \text { Isolate } \sqrt{x+7} \\
(\sqrt{x+7})^{2} & =(-1+\sqrt{x+14})^{2} & \text { Square both sides. } \\
x+7 & =1-2 \sqrt{x+14}+(\sqrt{x+14})^{2} \\
x+7 & =1-2 \sqrt{x+14}+x+14 \\
2 \sqrt{x+14} & =8 & \\
\sqrt{x+14} & =4 & \text { Square again. } \\
x+14 & =16 & \text { Subtract } 14 \text { from both sides } \\
x & =2 & \text { Square again. }
\end{array}
$$

As in method 1, check your answer(s).
Which method is simpler?
Example 4:
Solve $\sqrt{2 x+9}-\sqrt{x+5}=2$.

## Solution:

$$
\begin{array}{rll}
\sqrt{2 x+9}-\sqrt{x+5} & =2 & \text { Isolate a square root. } \\
\sqrt{2 x+9} & =2+\sqrt{x+5} & \text { Square both sides. } \\
(\sqrt{2 x+9})^{2} & =(2+\sqrt{x+5})^{2} & \\
2 x+9 & =4+4 \sqrt{x+5}+x+5 \\
x & =4 \sqrt{x+5} & \text { Square both sides again. } \\
x^{2} & =16(x+5) & \\
x^{2} & =16 x+80 & \\
x^{2}-16 x-80 & =0 & \\
(x-20)(x+4) & =0 &
\end{array}
$$

There are two solutions:
$\boldsymbol{x}_{1}=\mathbf{2 0}$ which needs to be checked.

$$
\begin{array}{r}
\sqrt{2(20)+9}-\sqrt{20+5}=2 \\
\sqrt{49}-\sqrt{25}=2 \\
7-5=2
\end{array}
$$

Thus $\boldsymbol{x}_{1}=\mathbf{2 0}$ is a solution.
Now check $\boldsymbol{x}_{2}=-4$.

$$
\begin{aligned}
\sqrt{2(-4)+9}-\sqrt{-4+5} & =2 \\
\sqrt{1}-\sqrt{1} & =2 \\
0 & =2 \quad \text { which is false. }
\end{aligned}
$$

Thus $x_{2}=-4$ is extraneous.

### 64.4 Exercise 64

1. Solve $2 \sqrt{3 x-7}-11=0$.
2. Solve $\sqrt{13 x}+39=13$.
3. Solve $\sqrt{\boldsymbol{x}+\mathbf{8 0}}-\sqrt{\boldsymbol{x}-\mathbf{5}}-\mathbf{5}=\mathbf{0}$.
4. Solve $\sqrt{3 x+13}-\sqrt{x-1}=4$.

## STOP!

1. Solve $2 \sqrt{3 x-7}-11=0$.

## Solution:

$$
\begin{aligned}
2 \sqrt{3 x-7}-11 & =0 \quad \text { Isolate the square root term. } \\
2 \sqrt{3 x-7} & =11 \quad \text { Square both sides. } \\
4(\sqrt{3 x-7})^{2} & =121 \\
4(3 x-7) & =121 \\
3 x-7 & =\frac{121}{4} \\
3 x & =\frac{121}{4}+\frac{7}{1} \cdot \frac{4}{4} \\
3 x & =\frac{121+28}{4} \\
x & =\frac{149}{3 \cdot 4}
\end{aligned}
$$

Now check $x=\frac{149}{12}$

$$
\begin{aligned}
2 \sqrt{3\left(\frac{149}{12}\right)-7}-11 & =0 \\
2 \sqrt{\frac{149}{4}-\frac{28}{4}}-11 & =0 \\
2 \sqrt{\frac{121}{4}}-11 & =0 \\
2\left(\frac{11}{2}\right)-11 & =0 \\
11-11 & =0
\end{aligned}
$$

So $x=\frac{149}{12}$ is a (the) solution.
2. Solve $\sqrt{13 x}+39=13$.

## Solution:

$$
\begin{aligned}
\sqrt{13 x}+39 & =13 \quad \text { Isolate the square root term. Subtract } 39 \text { from both sides. } \\
\sqrt{13 x} & =-\mathbf{2 6} \quad \text { You should stop here. If you don't, square both sides. } \\
\mathbf{1 3 x} & =\mathbf{6 7 6} \\
\boldsymbol{x} & =\mathbf{5 2}
\end{aligned}
$$

Now check $\boldsymbol{x}=\mathbf{5 2}$.

$$
\begin{aligned}
\sqrt{13 x}+39 & =13 \\
\sqrt{13(52)}+39 & =13 \\
26+39 & =13 \quad \text { which is clearly incorrect. }
\end{aligned}
$$

So $\boldsymbol{x}=\mathbf{5 2}$ is extraneous. We should have stopped as soon as we realized that the square root was negative.
3. Solve $\sqrt{x+80}-\sqrt{x-5}-5=0$.

## Solution:

$$
\begin{array}{rlrl}
\sqrt{x+80}-\sqrt{x-6}-5 & =0 & & \text { Add } 5 \text { to both sides. } \\
\sqrt{x+80}-\sqrt{x-5} & =5 & \text { Isolate a square root. } \\
\sqrt{x+80} & =5+\sqrt{x-5} & \text { Square both sides. } \\
(\sqrt{x+80})^{2} & =(5+\sqrt{x-5})^{2} \\
x+80 & =25+2(5) \sqrt{x-5}+x-5 \\
80 & =20+2(5) \sqrt{x-5} \\
60 & =10 \sqrt{x-5} & \\
6 & =\sqrt{x-5} & \text { Square both sides. } \\
(\sqrt{x-5})^{2} & =6^{2} \\
x-5 & =36 \\
x & =41
\end{array}
$$

Now check:

$$
\begin{array}{r}
\sqrt{x+80}-\sqrt{x-5}-5=0=0 \\
\sqrt{41+80}-\sqrt{41-5}-5=0=0 \\
\sqrt{121}-\sqrt{36}-5=0=0 \\
11-6-5=0=0
\end{array}
$$

Thus $x=41$ is a (the) solution.
4. Solve $\sqrt{3 x+13}-\sqrt{x-1}=4$.

Solution:

$$
\begin{array}{rll}
\sqrt{3 x+13}-\sqrt{x-1} & =4 \\
\sqrt{3 x+13} & =4+\sqrt{x-1} \quad \text { Isolate a square root. } \\
(\sqrt{3 x+13})^{2} & =(4+\sqrt{x-1})^{2} \quad \text { Square both sides. } \\
3 x+13 & =16+2(4) \sqrt{x-1}+x-1 \\
2 x+13 & =15+8 \sqrt{x-1} \\
2 x-2 & =8 \sqrt{x-1} \quad \text { Factor } 2 . \text { Then divide by } 2 . \\
(x-1)^{2} & =(4 \sqrt{x-1})^{2} \\
x^{2}-2 x+1 & =16(x-1) \\
x^{2}-2 x+1 & =16 x-16 \\
x^{2}-18 x+17 & =0 \\
(x-1)(x-17) & =0
\end{array}
$$

There are two solutions:
$x_{1}=17$ which needs to be checked.

$$
\begin{array}{r}
\sqrt{3 x+13}-\sqrt{x-1}=4 \\
\sqrt{3(17)+13}-\sqrt{17-1}=4 \\
\sqrt{51+13}-\sqrt{16}=4 \\
\sqrt{64}-4=4 \\
8-4=4
\end{array}
$$

Thus $x_{1}=17$ is a solution.
Now check $\boldsymbol{x}_{2}=\mathbf{1}$.

$$
\begin{aligned}
\sqrt{3 x+13}-\sqrt{x-1} & =4 \\
\sqrt{3(1)+13}-\sqrt{1-1} & =4 \\
\sqrt{16}-\sqrt{0} & =4 \\
4-0 & =4
\end{aligned}
$$

Thus $\boldsymbol{x}_{2}=1$ is also a solution.

## Chapter 65

# Application Problems Using Right Triangles and/or Radicals 

### 65.1 Youtube

https://www.youtube.com/playlist?list=PLB56CE348F462CBBA\&feature=view_all

### 65.2 Basics

In problems involving square roots, isolate a square root, square both sides, solve, check your answer(s).

### 65.3 Examples

## Example 1:

A pedestrian walks at the rate of the square root of a number (in mph). A bicyclist's speed is the square root of the sum of three times that number and 16. The sum of the distances traveled by the pedestrian in $\mathbf{3}$ hours and the bicyclist in $\mathbf{2}$ hours is $\mathbf{2 8}$ miles. Find the speed of the pedestrian and that of the bicyclist.

## Solution:

A pedestrian walks at the rate of the square root of a number $\boldsymbol{x}$ (in mph ) which means $\sqrt{\boldsymbol{x}}$.
A bicyclist's speed is the square root of the sum of three times that number $\boldsymbol{x}$ and $\mathbf{1 6}$. Speed $=\sqrt{\mathbf{3 x}+\mathbf{1 6}}$.

|  | Rate | -Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Pedestrian | $\sqrt{x}$ | 3 | $3 \sqrt{x}$ |
| Bicyclist | $\sqrt{3 x+16}$ | 2 | $2 \sqrt{3 x+16}$ |
| $\begin{aligned} 3 \sqrt{x}+2 \sqrt{3 x+16} & =28 \\ 2 \sqrt{3 x+16} & =28-3 \sqrt{x} \\ (2 \sqrt{3 x+16})^{2} & =(28-3 \sqrt{x})^{2} \end{aligned}$ |  |  |  |
|  |  |  |  |
|  |  |  |  |
| $4(3 x+16)=28^{2}-2(28)(3) \sqrt{x}+(3 \sqrt{x})^{2}$ |  |  |  |
| $12 x+64=784-168 \sqrt{x}+9 x$ |  |  |  |
| $3 x-720=-168 \sqrt{x}$ |  |  |  |
| $3(x-240)=-3(56) \sqrt{x}$ |  |  |  |
| $x-240=-56 \sqrt{x}$ |  |  |  |
| $(x-240)^{2}=(-56 \sqrt{x})^{2}$ |  |  |  |
| $x^{2}-480 x+57600=3136 x$ |  |  |  |
| $x^{2}-3616 x+57600=0$ |  |  |  |
| $\begin{aligned} P & =57,600 \\ S & =-3616 \end{aligned}$ |  |  |  |
| -1 \| |  |  |  |
| -2 \|l $\quad-28,800$ |  |  |  |
| -3 \| -19,200 |  |  |  |
| -4 \| -14,400 |  |  |  |
| -5 $-11,520$ |  |  |  |
| -6 \| $\quad-9,600$ |  |  |  |
| -8 \| $\quad-7,200$ |  |  |  |
| -9 \| -6,400 |  |  |  |
| -10 $-5,760$ |  |  |  |
| -12 \| -4,800 |  |  |  |
| -15 $-3,840$ |  |  |  |
| -16 \| $\quad-3,600$ sum $=-3,616$ |  |  |  |
| $x^{2}-16 x-3600 x+57600=0$ |  |  |  |
| $x(x-16)-3600(x-16)=0$ |  |  |  |
| $(x-16)(x-3600)=0$ |  |  |  |

Whether $\boldsymbol{x}=\mathbf{3 , 6 0 0} \mathrm{mph}$ is extraneous or not, it makes no sense in this problem (unless the pedestrian is the six million dollar-man and the bicyclist is superwoman.)

Check $\boldsymbol{x}=16$ :

$$
\begin{aligned}
3 \sqrt{x}+2 \sqrt{3 x+16} & =28 \\
3 \sqrt{16}+2 \sqrt{3(16)+16} & =28 \\
3 \cdot 4+2 \sqrt{64} & =28 \\
12+16 & =28
\end{aligned}
$$

The pedestrian's speed is $\sqrt{\mathbf{1 6}}=\mathbf{4} \mathrm{mph}$ and the bicyclist's speed is $\sqrt{\mathbf{3 ( 1 6 )}+\mathbf{1 6}}=\mathbf{8} \mathrm{mph}$.
Example 2:
A six-ft man casts a shadow that is the square root of a number. At the same time on the same level ground an eighteen-ft pole casts a shadow that is the square root of the sum of six times that number and 75. How long is the shadow cast by the man?

$$
\begin{aligned}
& \text { Solution: } \\
& \frac{\text { His shadow }}{\text { Man }}=\frac{\text { Its shadow }}{\text { Pole }} \\
& \frac{\sqrt{x}}{6}=\frac{\sqrt{6 x+75}}{18} \\
& \frac{\sqrt{x}}{1}=\frac{\sqrt{6 x+75}}{3} \\
& 3 \sqrt{x}=\sqrt{6 x+75} \\
& (3 \sqrt{x})^{2}=(\sqrt{6 x+75})^{2} \\
& 9 x=6 x+75 \\
& 3 x=75 \\
& x=25
\end{aligned}
$$

The man's shadow is $\sqrt{\mathbf{2 5}}=\mathbf{5} \mathrm{ft}$. Check:

$$
\begin{aligned}
\frac{\sqrt{x}}{6} & =\frac{\sqrt{6 x+75}}{18} \\
\frac{\sqrt{25}}{6} & =\frac{\sqrt{6(25)+75}}{18} \\
\frac{5}{6} & =\frac{\sqrt{150+75}}{18} \\
\frac{5}{1} & =\frac{\sqrt{225}}{3} \Rightarrow \frac{5}{1}=\frac{15}{3}
\end{aligned}
$$

Example 3:
A $\mathbf{2 5}$-ft ladder is used to clean a gutter on top of a wall. The foot of the ladder is $\boldsymbol{x}$ feet from the foot of the wall. The top of the ladder reaches the top of the wall. This height is the sum of $\mathbf{3} \boldsymbol{x}$ and three. Find the height of the wall.

## Solution:

25 is the hypotenuse of a right triangle whose legs are $\boldsymbol{x}$ and $\mathbf{3 x}+\mathbf{3}$ respectively.

$$
\begin{aligned}
& x^{2}+(3 x+3)^{2}=25^{2} \\
& x^{2}+9 x^{2}+18 x+9=625 \\
& 10 x^{2}+18 x-616=0 \\
& 2\left(5 x^{2}+9 x-308\right)=0 \\
& \begin{array}{rlll}
P & = & -1540 \\
S & = & \\
\hline-1 & 1,540 & \\
-2 & 770 \\
-4 & 385 & \\
-5 & 308 & \\
-7 & 220 & \\
-10 & 154 & \\
-11 & 140 & \\
-20 & 77 & \\
-22 & 70 & \text { sum }=-35+44=9
\end{array} \\
& 5 x^{2}+9 x-308=0 \\
& 5 x^{2}-35 x+44 x-308=0 \\
& 5 x(x-7)+44(x-7)=0 \\
& (x-7)(5 x+44)=0
\end{aligned}
$$

Whether $\boldsymbol{x}=-\frac{\mathbf{4 4}}{\mathbf{5}}$ is extraneous or not is immaterial. A negative number cannot be a height.
Check $\boldsymbol{x}=\mathbf{7}$ :

$$
\begin{aligned}
x^{2}+(3 x+3)^{2} & =25^{2} \\
7^{2}+(3 \cdot 7+3)^{2} & =25^{2} \\
49+24^{2} & =625 \\
49+576 & =625
\end{aligned}
$$

The height of the wall is $\mathbf{3} \cdot \mathbf{7}+\mathbf{3}=\mathbf{2 4} \mathrm{ft}$.
Example 4:
The number of hours a painter takes to paint a room alone is the square root of a certain number. His assistant takes $\mathbf{1}$ more hour to paint the same room alone. Working together, they take $\mathbf{1} \frac{\mathbf{1}}{\mathbf{6}}$ hours. How long does it take the painter working alone?

## Solution:

The painter takes $\sqrt{\boldsymbol{x}}$ hours to paint the room. His assistant takes $1+\sqrt{\boldsymbol{x}}$ hours.

|  | Time to complete painting | Part of job in one hour |
| :---: | :---: | :---: |
| Painter | $\sqrt{\boldsymbol{x}}$ | $\frac{1}{\sqrt{x}}$ |
| Assistant | $\mathbf{1}+\sqrt{\boldsymbol{x}}$ | $\frac{\mathbf{1}}{\mathbf{1}+\sqrt{\boldsymbol{x}}}$ |
| Together | $\mathbf{1} \frac{\mathbf{1}}{\mathbf{6}}=\frac{\mathbf{6}}{\mathbf{5}}$ | $\frac{\mathbf{5}}{\mathbf{6}}$ |

The part of the room painted by the painter in one hour added to the part done by his assistant in one hour equals the part they complete working together in one hour.

$$
\begin{aligned}
\frac{1}{\sqrt{x}}+\frac{1}{1+\sqrt{x}} & =\frac{5}{6} \\
\frac{6(\sqrt{x})(1+\sqrt{x})}{\sqrt{x}}+\frac{6(\sqrt{x})(1+\sqrt{x})}{1+\sqrt{x}} & =\frac{5 \cdot 6(\sqrt{x})(1+\sqrt{x})}{6} \\
6(1+\sqrt{x})+6 \sqrt{x} & =5(\sqrt{x})(1+\sqrt{x}) \\
6+6 \sqrt{x}+6 \sqrt{x} & =5 \sqrt{x}+5 x \\
6+(12-5) \sqrt{x} & =5 x \\
7 \sqrt{x} & =5 x-6 \\
(7 \sqrt{x})^{2} & =(5 x-6)^{2} \\
49 x & =25 x^{2}-(2)(5 x)(6)+6^{2} \\
0 & =25 x^{2}-60 x-49 x+36 \\
0 & =25 x^{2}-109 x+36
\end{aligned}
$$

$$
\begin{aligned}
& P=900 \\
& S=-109 \\
& \begin{array}{l|l}
-1 & -900 \\
-2 & -450
\end{array} \\
& \begin{array}{l|l}
-3 & -300
\end{array} \\
& \begin{array}{l|l}
-4 & -225
\end{array} \\
& \begin{array}{l|l}
-5 & -180
\end{array} \\
& \begin{array}{l|l}
-6 & -150
\end{array} \\
& -9 \mid-100 \text { sum }=-9-100=-109 \\
& 25 x^{2}-109 x+36=0 \\
& 25 x^{2}-9 x-100 x+36=0 \\
& x(25 x-9)-4(25 x-9)=0 \\
& (25 x-9)(x-4)=0
\end{aligned}
$$

Check $x=\frac{9}{25}$

$$
\frac{1}{\sqrt{x}}+\frac{1}{1+\sqrt{x}}=\frac{5}{6}
$$

$\frac{1}{\sqrt{\frac{9}{25}}}+\frac{1}{1+\sqrt{\frac{9}{25}}}=\frac{5}{6}$

$$
\frac{1}{\frac{3}{5}}+\frac{1}{1+\frac{3}{5}}=\frac{5}{6}
$$

$$
\frac{5}{3}+\frac{1}{\frac{8}{5}}=\frac{5}{6}
$$

$$
\frac{5 \cdot 8}{3 \cdot 8}+\frac{5 \cdot 3}{8 \cdot 3}=\frac{5}{6}
$$

$$
\frac{5 \cdot 8}{3 \cdot 8}+\frac{5 \cdot 3}{8 \cdot 3}=\frac{5}{6}
$$

$$
\frac{55}{24} \neq \frac{5}{6}
$$

$x=\frac{9}{25}$ is not a solution.
Check $\boldsymbol{x}=4$ :

$$
\begin{aligned}
\frac{1}{\sqrt{x}}+\frac{1}{1+\sqrt{x}} & =\frac{5}{6} \\
\frac{1}{\sqrt{4}}+\frac{1}{1+\sqrt{4}} & =\frac{5}{6} \\
\frac{1}{2}+\frac{1}{1+2} & =\frac{5}{6} \\
\frac{3}{6}+\frac{2}{6} & =\frac{5}{6}
\end{aligned}
$$

Thus it takes the painter $\sqrt{\mathbf{4}}=2$ hours to paint the room by himself/herself.

### 65.4 Exercise 65

1. A pedestrian walks at the rate of the square root of the difference of a number (in mph) and 4. A bicyclist's speed is the square root of the sum of four times that number and $\mathbf{1}$. The sum of the distances traveled by the pedestrian in $\mathbf{4}$ hours and the bicyclist in $\mathbf{2}$ hours is $\mathbf{3 4}$ miles. Find the speed of the pedestrian and that of the bicyclist.
2. A five-ft man casts a shadow that is the square root of a number. At the same time on the same level ground a $\mathbf{2 0}$-ft pole casts a shadow that is the square root of the sum of $\mathbf{1 5}$ times that number and 16. How long is the shadow cast by the man?
3. A $\mathbf{3 7}$-ft ladder is used to clean a gutter on top of a wall. The foot of the ladder is the square root of a certain number $\boldsymbol{x}$ feet from the bottom of the wall. The top of the ladder reaches the top of the wall. This height is the square root of the sum of $\mathbf{8 x}$ and $\mathbf{7 3}$. Find the height of the wall.
4. The number of hours one painter takes to paint a room by himself is the square root of a certain number. His assistant takes two more hours to paint the same room alone. If they work together, it takes them $2 \frac{2}{5}$ hours. How long does it take the painter working alone?

## STOP!

1. A pedestrian walks at the rate of the square root of the difference of a number (in mph) and 4. A bicyclist's speed is the square root of the sum of four times that number and $\mathbf{1}$. The sum of the distances traveled by the pedestrian in $\mathbf{4}$ hours and the bicyclist in $\mathbf{2}$ hours is $\mathbf{3 4}$ miles. Find the speed of the pedestrian and that of the bicyclist.

## Solution:

A pedestrian walks at the rate of the square root of the difference of a number $\boldsymbol{x}$ (in mph ) and 4 means $\sqrt{\boldsymbol{x}-4}$.

A bicyclist's speed is the square root of the sum of four times that number $\boldsymbol{x}$ and $\mathbf{1}$.
Speed $=\sqrt{4 x+1}$.

|  | Rate | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Pedestrian | $\sqrt{x-4}$ | 4 | $4 \sqrt{x-4}$ |
| Bicyclist | $\sqrt{4 x+1}$ | 2 | $2 \sqrt{4 x+1}$ |

$$
\begin{aligned}
4 \sqrt{x-4}+2 \sqrt{4 x+1} & =34 \\
2 \sqrt{4 x+1} & =34-4 \sqrt{x-4} \\
(2 \sqrt{4 x+1})^{2} & =(34-4 \sqrt{x-4})^{2} \\
4(4 x+1) & =34^{2}-2(34)(4) \sqrt{x-4}+(4 \sqrt{x-4})^{2} \\
16 x+4 & =1156-272 \sqrt{x-4}+16 x-64 \\
4+64-1156 & =-272 \sqrt{x-4} \\
-1088 & =-272 \sqrt{x-4} \\
4 & =\sqrt{x-4} \\
16 & =x-4 \\
x & =20
\end{aligned}
$$

Check $\boldsymbol{x}=\mathbf{2 0}$ :

$$
\begin{aligned}
4 \sqrt{x-4}+2 \sqrt{4 x+1} & =34 \\
4 \sqrt{20-4}+2 \sqrt{4(20)+1} & =34 \\
4 \sqrt{16}+2 \sqrt{81} & =34 \\
16+2(9) & =34
\end{aligned}
$$

The pedestrian's speed is $\sqrt{20-4}=4 \mathrm{mph}$ and the bicyclist's speed is $\sqrt{4(20)+1}=9 \mathrm{mph}$.
2. A five-ft man casts a shadow that is the square root of a number. At the same time on the same level ground a $20-\mathrm{ft}$ pole casts a shadow that is the square root of the sum of $\mathbf{1 5}$ times that number and 16. How long is the shadow cast by the man?

## Solution:

$$
\begin{aligned}
\frac{\text { His shadow }}{\text { Man }} & =\frac{\text { Its shadow }}{\text { Pole }} \\
\frac{\sqrt{x}}{5} & =\frac{\sqrt{15 x+16}}{20} \\
\frac{\sqrt{x}}{1} & =\frac{\sqrt{15 x+16}}{4} \\
4 \sqrt{x} & =\sqrt{15 x+16} \\
(4 \sqrt{x})^{2} & =(\sqrt{15 x+16})^{2} \\
16 x & =15 x+16 \\
x & =16
\end{aligned}
$$

The man's shadow is $\sqrt{\mathbf{1 6}}=\mathbf{4} \mathrm{ft}$.
Check:

$$
\begin{aligned}
\frac{\sqrt{x}}{5} & =\frac{\sqrt{15 x+16}}{20} \\
\frac{\sqrt{16}}{5} & =\frac{\sqrt{15(16)+16}}{20} \\
\frac{4}{5} & =\frac{\sqrt{15(16)+16}}{20} \\
\frac{4}{1} & =\frac{\sqrt{256}}{4} \\
\frac{4}{1} & =\frac{16}{4}
\end{aligned}
$$

3. A $\mathbf{3 7}$ - ft ladder is used to clean a gutter on top of a wall. The foot of the ladder is the square root of a certain number $\boldsymbol{x}$ feet from the bottom of the wall. The top of the ladder reaches the top of the wall. This height is the square root of the sum of $\mathbf{8 x}$ and $\mathbf{7 3}$. Find the height of the wall.

## Solution:

25 is the hypotenuse of a right triangle whose legs are $\sqrt{\boldsymbol{x}}$ and $\sqrt{\mathbf{8 x + 7 3}}$ respectively.

$$
\begin{aligned}
(\sqrt{x})^{2}+(\sqrt{8 x+73})^{2} & =37^{2} \\
x+8 x+73 & =1369 \\
9 x & =1296 \\
x & =144
\end{aligned}
$$

Check $x=144$ :

$$
\begin{aligned}
(\sqrt{x})^{2}+(\sqrt{8 x+73})^{2} & =37^{2} \\
(\sqrt{144})^{2}+(\sqrt{8 \cdot 144+73})^{2} & =37^{2} \\
144+8 \cdot 144+73 & =1369 \\
144+1152+73 & =1369 \\
1369 & =1369
\end{aligned}
$$

The wall is $\sqrt{\mathbf{8 \cdot 1 4 4 + 7 3}}=\sqrt{\mathbf{1 2 2 5}}=\mathbf{3 5} \mathrm{ft}$ high.
4. The number of hours one painter takes to paint a room by himself is the square root of a certain number. His assistant takes two more hours to paint the same room alone. If they work together, it takes them $2 \frac{2}{5}$ hours. How long does it take the painter working alone?

## Solution:

The painter takes $\sqrt{\boldsymbol{x}}$ hours to paint the room. His assistant takes $2+\sqrt{\boldsymbol{x}}$ hours.

|  | Time to complete painting | Part of job in one hour |
| :---: | :---: | :---: |
| Painter | $\sqrt{x}$ | $\frac{1}{\sqrt{x}}$ |
| Assistant | $\mathbf{1}+\sqrt{\boldsymbol{x}}$ | $\frac{\mathbf{1}}{2+\sqrt{x}}$ |
| Together | $\mathbf{2} \frac{\mathbf{2}}{\mathbf{5}}=\frac{\mathbf{1 2}}{\mathbf{5}}$ | $\frac{\mathbf{5}}{\mathbf{1 2}}$ |

The part of the room painted by the painter in one hour added to the part done by his assistant in one hour equals the part they complete working together in one hour.

$$
\begin{aligned}
\frac{1}{\sqrt{x}}+\frac{1}{2+\sqrt{x}} & =\frac{5}{12} \\
\frac{12(\sqrt{x})(2+\sqrt{x})}{\sqrt{x}+\frac{12(\sqrt{x})(2+\sqrt{x})}{2+\sqrt{x}}} & =\frac{5 \cdot 12(\sqrt{x})(2+\sqrt{x})}{12} \\
12(2+\sqrt{x})+12 \sqrt{x} & =5(\sqrt{x})(2+\sqrt{x}) \\
24+12 \sqrt{x}+12 \sqrt{x} & =10 \sqrt{x}+5 x \\
24+(12+12-10) \sqrt{x} & =5 x \\
24+14 \sqrt{x} & =5 x \\
14 \sqrt{x} & =5 x-24 \\
(14 \sqrt{x})^{2} & =(5 x-24)^{2} \\
196 x & =25 x^{2}-(2)(5 x)(24)+24^{2} \\
196 x & =25 x^{2}-240 x+576 \\
0 & =25 x^{2}-436 x+576
\end{aligned}
$$

$$
P=14400
$$

$$
\begin{array}{r|rl}
S=-436 & \\
\hline-1 & -14400 \quad \text { fast forward }
\end{array}
$$

$$
\begin{array}{c|cc}
-10 & -1440 & \text { fast forward }
\end{array}
$$

$$
\begin{array}{l|ll}
-20 & -720 & \text { fast forward }
\end{array}
$$

$$
\begin{array}{l|ll}
-30 & -480 & \text { fast forward }
\end{array}
$$

$$
\begin{array}{l|ll}
-36 & -400 & \text { sum }=-436
\end{array}
$$

$$
25 x^{2}-436 x+576=0
$$

$$
25 x^{2}-36 x-400 x+576=0
$$

$$
x(25 x-36)-16(25 x-36)=0
$$

$$
(25 x-36)(x-16)=0
$$

Check:
$x=\frac{36}{25}$
$\frac{1}{\sqrt{x}}+\frac{1}{2+\sqrt{x}}=\frac{5}{12}$
$\frac{1}{\sqrt{\frac{36}{25}}}+\frac{1}{2+\sqrt{\frac{36}{25}}}=\frac{5}{12}$
$\frac{1}{\frac{6}{5}}+\frac{1}{2+\frac{6}{5}}=\frac{5}{12}$
$\frac{5}{6}+\frac{2}{\frac{16}{5}}=\frac{5}{12}$
$\frac{5 \cdot 8}{6 \cdot 8}+\frac{5 \cdot 3}{16 \cdot 3}=\frac{5}{12}$

$$
\frac{5 \cdot 8}{6 \cdot 8}+\frac{5 \cdot 3}{16 \cdot 3}=\frac{5}{12}
$$

$$
\frac{55}{48} \neq \frac{5}{12}
$$

$x=\frac{36}{25}$ is not a solution.
Check $\boldsymbol{x}=16$ :

$$
\begin{aligned}
\frac{1}{\sqrt{x}}+\frac{1}{2+\sqrt{x}} & =\frac{5}{12} \\
\frac{1}{\sqrt{16}}+\frac{1}{2+\sqrt{16}} & =\frac{5}{12} \\
\frac{1}{4}+\frac{1}{2+4} & =\frac{5}{12} \\
\frac{3}{12}+\frac{2}{12} & =\frac{5}{12}
\end{aligned}
$$

Thus it takes the painter alone $\sqrt{\mathbf{1 6}}=\mathbf{4}$ hours to paint the room.

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## Chapter 66

## Higher Roots and Radical Expressions

### 66.1 Youtube

https://www.youtube.com/playlist?list=PLFCDD4A4002CC48C1\&feature=view_all

### 66.2 Basics

Recall that $\sqrt{\mathbf{6 4}}=\sqrt{\mathbf{8}^{2}}=\sqrt{\mathbf{8 \cdot 8}}=8 \geq 0$ because $\mathbf{8}^{\mathbf{2}}=\mathbf{6 4}$.
Now $\sqrt[3]{\mathbf{6 4}}=\sqrt[3]{4^{3}}=\sqrt[3]{4 \cdot 4 \cdot 4}=4$ because $4^{3}=64$.
Similarly $\sqrt[4]{\mathbf{8 1}}=\sqrt[4]{\mathbf{3}^{4}}=\sqrt[4]{\mathbf{3 \cdot 3 \cdot 3 \cdot 3}}=\mathbf{3}$ because $\mathbf{3}^{4}=\mathbf{8 1} \geq \mathbf{0}$.
$\sqrt[5]{32}=\sqrt[5]{2^{5}}=\sqrt[5]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=2$ because $2^{5}=32$.
$\sqrt[6]{64}=\sqrt[6]{2^{6}}=\sqrt[6]{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=2$ because $2^{6}=64 \geq 0$.
Note in general $\sqrt[n]{\boldsymbol{a}^{\boldsymbol{n}}}=\sqrt[n]{\underbrace{\boldsymbol{a} \boldsymbol{a} \cdot \boldsymbol{a} \cdot \boldsymbol{a}}_{\boldsymbol{n} \text { factors } \boldsymbol{a}}}=\boldsymbol{a} \quad(\geq \mathbf{0}$ if $\boldsymbol{n}$ is even)
because $\boldsymbol{a}^{\boldsymbol{n}}=$ the radicand (the quantity under the radical sign).

### 66.3 Examples

Example 1:
Simplify $\sqrt[3]{-81 a^{6} b^{7} c^{8}}$

## Solution:

$$
\begin{aligned}
\sqrt[3]{-81 a^{6} b^{7} c^{8}} & =\sqrt[3]{(-3)^{3} \cdot 3\left(\mathrm{a}^{2}\right)^{3}\left(\mathrm{~b}^{2}\right)^{3} \cdot b\left(\mathrm{c}^{2}\right)^{3} \cdot c^{2}} \\
& =-3 \mathrm{a}^{2} \mathrm{~b}^{2} \mathrm{c}^{2} \sqrt[3]{3 b c^{2}}
\end{aligned}
$$

Example 2:


## Solution:

$$
\begin{aligned}
\sqrt[4]{81 a^{6} b^{7} c^{8}(x-9)^{9}} & =\sqrt[4]{3^{4} a^{4} \cdot a^{2} b^{4} \cdot b^{3}\left(c^{2}\right)^{4}\left[(x-9)^{2}\right]^{4}(x-9)} \\
& =3 a^{2} c^{2}(x-9)^{2} \sqrt[4]{a^{2} b^{3}(x-9)}
\end{aligned}
$$

Why is there no need to assume $\boldsymbol{b} \geq \mathbf{0}$ ?
Answer: If $\boldsymbol{b}<\mathbf{0}$ then $\sqrt[4]{\boldsymbol{b}^{\mathbf{7}}}$ would not be defined and the given problem would not be real.
Why is there no need to assume $\boldsymbol{c} \geq \mathbf{0}$ ?
Answer: $\boldsymbol{c}^{\mathbf{2}} \geq \mathbf{0}$ even if $\boldsymbol{c}<\mathbf{0}$ in front of the square root sign.
Example 3:
Simplify $\sqrt[5]{\frac{a^{24}(b+4)^{21}}{a^{9}(b+4)^{6}}}$

## Solution:

$$
\begin{aligned}
\sqrt[5]{\frac{a^{24}(b+4)^{21}}{a^{9}(b+4)^{6}}} & =\sqrt[5]{\frac{a^{24-9}(b+4)^{21-6}}{1}} \\
& =\sqrt[5]{a^{15}(b+4)^{15}} \\
& =\sqrt[5]{\left(a^{3}\right)^{5}\left[(b+4)^{3}\right]^{5}} \\
& =a^{3}(b+4)^{3}
\end{aligned}
$$

Example 4:
Simplify $5 a \sqrt[3]{2 a^{2}}+\sqrt[4]{32 a^{6}}-3 \sqrt[5]{2^{6} a^{7}}$. Assume $a \geq 0$.

## Solution:

$$
\begin{aligned}
5 a \sqrt[3]{2 a^{2}}+\sqrt[4]{32 a^{6}}-3 \sqrt[5]{2^{6} a^{7}} & =5 a \sqrt[3]{2 a^{2}}+\sqrt[4]{2^{4}(2) a^{4} a^{2}}-3 \sqrt[5]{2^{5}(2) a^{5} a^{2}} \\
& =5 a \sqrt[3]{2 a^{2}}+2 a \sqrt[4]{(2) a^{2}}-3 \cdot 2 a \sqrt[5]{(2) a^{2}} \\
& =5 a \sqrt[3]{2 a^{2}}+2 a \sqrt[4]{2 a^{2}}-6 a \sqrt[5]{2 a^{2}} \\
& =a\left(5 \sqrt[3]{2 a}+2 \sqrt[4]{2 a^{2}}-6 \sqrt[5]{2 a^{2}}\right)
\end{aligned}
$$

Too bad the indices 3, 4, and $\mathbf{5}$ are not the same.
Example 5:
Simplify $\frac{\sqrt[6]{(2 x-1)^{7}}}{\sqrt[3]{(2 x-1)^{4}}}$. Assume $2 x>1$.

## Solution:

$$
\begin{aligned}
\frac{\sqrt[6]{(2 x-1)^{7}}}{\sqrt[3]{(2 x-1)^{4}}} & =\frac{\sqrt[6]{(2 x-1)^{6}(2 x-1)}}{\sqrt[3]{(2 x-1)^{3}(2 x-1)}} \\
& =\frac{(2 x-1) \sqrt[6]{(2 x-1)}}{(2 x-1) \sqrt[3]{(2 x-1)}} \\
& =\frac{\sqrt[6]{(2 x-1)}}{\sqrt[3]{(2 x-1)}} \quad \quad \text { Note: } \sqrt[3]{2^{3}}=\sqrt[6]{2^{6}} \text { and } \sqrt[3]{(2 x-1)}=\sqrt[6]{(2 x-1)^{2}} \\
& =\frac{\sqrt[6]{(2 x-1)}}{\sqrt[6]{(2 x-1)^{2}}} \\
& =\sqrt[6]{\frac{(2 x-1)}{(2 x-1)^{2}}}=\frac{1}{\sqrt[6]{2 x-1}}
\end{aligned}
$$

Example 6:
Rationalize $\frac{\boldsymbol{x}}{\sqrt[3]{\boldsymbol{x}+\mathbf{1}}}$. Assume $\boldsymbol{x}>\mathbf{- 1}$.

## Solution:

$$
\begin{aligned}
\frac{x}{\sqrt[3]{x+1}} & =\frac{x}{\sqrt[3]{x+1}} \cdot \frac{\sqrt[3]{(x+1)^{2}}}{\sqrt[3]{(x+1)^{2}}} \\
& =\frac{x \sqrt[3]{(x+1)^{2}}}{\sqrt[3]{(x+1)^{3}}} \\
& =\frac{x \sqrt[3]{(x+1)^{2}}}{x+1}
\end{aligned}
$$

### 66.4 Exercise 66

1. Simplify $\sqrt[5]{-64 a^{8} b^{10} c^{12}}$
2. Simplify $\sqrt[3]{81 a^{7} b^{9} c^{11}(x-9)^{5}}$
3. Simplify $\sqrt[4]{\frac{a^{24}(b+4)^{21}}{a^{9}(b+4)^{6}}}$
4. Simplify $5 a \sqrt[3]{2 a^{2}}+\sqrt[3]{16 a^{5}}-3 \sqrt[3]{2^{7} a^{5}}$.
5. Simplify $\frac{\sqrt[5]{(2 x-1)^{12}}}{\sqrt[10]{(2 x-1)^{16}}}$. Assume $2 x>1$.
6. Rationalize $\frac{x^{2}}{\sqrt[5]{(x+1)^{3}}}$. Assume $x>-1$.

## STOP!

1. Simplify $\sqrt[5]{-64 a^{8} b^{10} c^{12}}$

## Solution:

$$
\begin{aligned}
\sqrt[5]{-64 a^{8} b^{10} c^{12}} & =\sqrt[5]{-2^{5} \cdot 2 a^{5} a^{3}\left(b^{2}\right)^{5}\left(c^{2}\right)^{5} c^{2}} \\
& =-2 a b^{2}\left(c^{2}\right) \sqrt[5]{2 a^{3} c^{2}}
\end{aligned}
$$

2. Simplify $\sqrt[3]{81 a^{7} b^{9} c^{11}(x-9)^{5}}$

Solution:

$$
\begin{aligned}
\sqrt[3]{81 a^{7} b^{9} c^{11}(x-9)^{5}} & =\sqrt[3]{3^{3}(3)\left(\mathrm{a}^{2}\right)^{3} a\left(\mathrm{~b}^{3}\right)^{3}\left(c^{3}\right)^{3} c^{2}\left[(x-9)^{3}\right](x-9)} \\
& =3 \mathrm{a}^{2} \mathrm{~b}^{3} \mathrm{c}^{3}(x-9) \sqrt[3]{3 a c^{2}(x-9)}
\end{aligned}
$$

3. Simplify $\sqrt[4]{\frac{a^{24}(b+4)^{21}}{a^{9}(b+4)^{6}}}$

Solution:

$$
\begin{aligned}
\sqrt[4]{\frac{a^{24}(b+4)^{21}}{a^{9}(b+4)^{6}}} & =\sqrt[4]{\frac{a^{24-9}(b+4)^{21-6}}{1}} \\
& =\sqrt[4]{a^{15}(b+4)^{15}} \\
& =\sqrt[4]{\left(a^{3}\right)^{4} a^{3}\left[(b+4)^{3}\right]^{4}(b+4)^{3}} \\
& =a^{3}(b+4)^{3} \sqrt[4]{a^{3}(b+4)^{3}}
\end{aligned}
$$

4. Simplify $5 a \sqrt[3]{2 a^{2}}+\sqrt[3]{16 a^{5}}-3 \sqrt[3]{2^{7} a^{5}}$.

## Solution:

$$
\begin{aligned}
& 5 a \sqrt[3]{2 a^{2}}+\sqrt[3]{16 a^{5}}-3 \sqrt[3]{2^{7} a^{5}} \\
= & 5 a \sqrt[3]{2 a^{2}}+\sqrt[3]{2^{3}(2) a^{3} a^{2}}-3 \sqrt[3]{{\left(2^{2}\right)^{3}(2) a^{3} a^{2}}^{2}} \begin{aligned}
= & 5 a \sqrt[3]{2 a^{2}}+2 a \sqrt[3]{2 a^{2}}-(3)\left(2^{2}\right) a \sqrt[3]{2 a^{2}} \\
= & (5+2-12) a \sqrt[3]{2 a^{2}} \\
= & -5 a \sqrt[3]{2 a^{2}}
\end{aligned},=\frac{1}{}
\end{aligned}
$$

Too bad the indices of the square roots, $\mathbf{3}, \mathbf{4}$, and $\mathbf{5}$ are not the same.
5. Simplify $\frac{\sqrt[5]{(2 x-1)^{12}}}{\sqrt[10]{(2 x-1)^{16}}}$. Assume $2 x>1$.

Solution:

$$
\begin{aligned}
\frac{\sqrt[5]{(2 x-1)^{12}}}{\sqrt[10]{(2 x-1)^{16}}} & =\frac{\sqrt[5]{\left[(2 x-1)^{2}\right]^{5}(2 x-1)^{2}}}{\sqrt[10]{\left[(2 x-1)^{3}\right]^{5}(2 x-1)}} \\
& =\frac{(2 x-1)^{2} \sqrt[5]{(2 x-1)^{2}}}{(2 x-1)^{3} \sqrt[10]{(2 x-1)}} \\
& =\frac{\sqrt[10]{(2 x-1)^{4}}}{(2 x-1) \sqrt[10]{(2 x-1)}} \\
& =\frac{\sqrt[10]{(2 x-1)^{3}}}{2 x-1}
\end{aligned}
$$

6. Rationalize $\frac{x^{2}}{\sqrt[5]{(x+1)^{3}}}$.

Assume $\boldsymbol{x}>\mathbf{- 1}$.

## Solution:

$$
\begin{aligned}
\frac{x^{2}}{\sqrt[5]{(x+1)^{3}}} & =\frac{x^{2}}{\sqrt[5]{(x+1)^{3}}} \cdot \frac{\sqrt[5]{(x+1)^{2}}}{\sqrt[5]{(x+1)^{2}}} \\
& =\frac{x^{2} \sqrt[5]{(x+1)^{2}}}{\sqrt[5]{(x+1)^{5}}} \\
& =\frac{x^{2} \sqrt[5]{(x+1)^{2}}}{x+1}
\end{aligned}
$$

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## Chapter 67

## Solving Quadratic Equations Using the Square Root Property

### 67.1 Youtube

https://www.youtube.com/playlist?list=PL5BEA49B4CFB06EE3\&feature=view_all

### 67.2 Basics

We learned how to solve $\boldsymbol{x}^{\mathbf{2}}-\mathbf{2 5}=\mathbf{0}$ earlier. We factored the left side (if possible) and set each factor to 0 .

$$
\begin{array}{r}
x^{2}-25=0 \\
(x+5)(x-5)=0
\end{array}
$$

So $\boldsymbol{x}=\mathbf{- 5}$ or $\boldsymbol{x}=\mathbf{5}$.
We could have approached the solution of $\boldsymbol{x}^{\mathbf{2}}-\mathbf{2 5}=\mathbf{0}$ by using the square root property. The technique involves taking square roots of both sides of an equation.

Given $\boldsymbol{x}^{2}-\mathbf{2 5}=\mathbf{0}$, first write $\boldsymbol{x}^{2}=\mathbf{2 5}$
The square root of the left side is $\boldsymbol{x}$. The square root of the right side is positive $\mathbf{5}$. But since $\boldsymbol{x}$ is a variable, it can be positive or negative. $\boldsymbol{x}=5$ because $\boldsymbol{x}^{2}=(5)^{2}=\mathbf{2 5}$. Also $\boldsymbol{x}=\mathbf{- 5}$ because $x^{2}=(-5)^{2}=25$.

### 67.3 Examples

Example 1:
Solve $x^{2}+3 x-10=0$ by factoring.

## Solution:

$$
\begin{array}{r}
x^{2}-3 x-10=0 \\
(x-5)(x+2)=0
\end{array}
$$

So $\boldsymbol{x}=\mathbf{5}$ or $\boldsymbol{x}=\mathbf{- 2}$.
Example 2:
Use the square root property to solve $\mathbf{4} \boldsymbol{x}^{2}-\mathbf{2 0} \boldsymbol{x}+\mathbf{2 5}=\mathbf{0}$.

## Solution:

$$
\begin{aligned}
4 x^{2}-20 x+25 & =0 \\
(2 x)^{2}-2(2 x)(5)+(5)^{2} & =0 \\
(2 x-5)^{2} & =0
\end{aligned}
$$

So $\boldsymbol{x}=\frac{\mathbf{5}}{\mathbf{2}}$.
Given a second degree equation, a quadratic equation, we expect two roots (solutions). We call $\boldsymbol{x}=\frac{\mathbf{5}}{\mathbf{2}}$ a double root.

Example 3:
Solve $\boldsymbol{x}^{\mathbf{2}}+\mathbf{9}=\mathbf{0}$ by using the square root property.

## Solution:

$$
\begin{aligned}
x^{2}+9 & =0 \\
x^{2} & =-9
\end{aligned}
$$

So $\boldsymbol{x}=\sqrt{-\mathbf{3}}$ which is not a real number.
Example 4:
Solve $\frac{\frac{x}{2}}{3}=\frac{\frac{3}{x}}{\frac{x}{2}}$ by using the square root property.

## Solution:

$$
\begin{aligned}
\frac{\frac{x}{2}}{3} & =\frac{3}{\frac{x}{2}} \\
\left(\frac{x}{2}\right)^{2} & =3^{2} \\
\frac{x}{2} & = \pm 3 \\
x & = \pm 6
\end{aligned}
$$

Example 5:
Solve $(x+9)^{2}=25$ by using the square root property.

## Solution:

$$
\begin{aligned}
(x+9)^{2} & =25 \\
\sqrt{(x+9)^{2}} & = \pm \sqrt{25} \\
x+9 & = \pm 5 \\
x & =-9 \pm 5
\end{aligned}
$$

Thus $x=-9+5=-4$ or $x=-9-5=-14$.
Example 6:
Solve $49(x+9)^{2}-121=0$ by using the square root property.

## Solution:

$$
\begin{aligned}
49(x+9)^{2}-121 & =0 \\
49(x+9)^{2} & =121 \\
\sqrt{49(x+9)^{2}} & = \pm \sqrt{121} \\
7(x+9) & = \pm 11 \\
x+9 & = \pm \frac{11}{7} \\
x & =-9 \pm \frac{11}{7}
\end{aligned}
$$

Thus $x=\frac{-63}{7}+\frac{11}{7}=-\frac{52}{7}$
or
$x=\frac{-63}{7}-\frac{11}{7}=-\frac{74}{7}$.

### 67.4 Exercise 67

1. Solve $x^{2}-7 x+12=0$ by factoring.
2. Use the square root property to solve $9 \boldsymbol{x}^{2}-42 x+49=0$.
3. Solve $\boldsymbol{x}^{\mathbf{2}}+\mathbf{8 1}=\mathbf{0}$ by using the square root property.
4. Solve $\frac{\frac{\boldsymbol{x}}{\mathbf{6}}}{\mathbf{9}}=\frac{\mathbf{9}}{\frac{\boldsymbol{x}}{\mathbf{6}}}$ by using the square root property.
5. Solve $(\boldsymbol{x}-\mathbf{1 2})^{\mathbf{2}}=\mathbf{1 0 0}$ by using the square root property.
6. Solve $\mathbf{6 4}(\boldsymbol{x}-\mathbf{5})^{\mathbf{2}} \mathbf{- 2 5}=\mathbf{0}$ by using the square root property.

## STOP!

1. Solve $\boldsymbol{x}^{2}-7 x+12=0$ by factoring.

## Solution:

$$
\begin{array}{r}
x^{2}-7 x+12=0 \\
(x-4)(x-3)=0
\end{array}
$$

So $\boldsymbol{x}=\mathbf{3}$ or $\boldsymbol{x}=4$.
2. Use the square root property to solve $\mathbf{9} \boldsymbol{x}^{2}-42 x+49=0$.

## Solution:

$$
\begin{aligned}
9 x^{2}-42 x+49 & =0 \\
(3 x)^{2}-2(3 x)(7)+(7)^{2} & =0 \\
(3 x-7)^{2} & =0
\end{aligned}
$$

So $\boldsymbol{x}=\frac{\mathbf{7}}{\mathbf{3}}$.
Given a second degree equation, a quadratic equation, we expect two roots (solutions). We call $x=\frac{\mathbf{7}}{\mathbf{3}}$ a double root.
3. Solve $\boldsymbol{x}^{\mathbf{2}}+\mathbf{8 1}=\mathbf{0}$ by using the square root property.

## Solution:

$$
\begin{aligned}
x^{2}+81 & =0 \\
x^{2} & =-81
\end{aligned}
$$

So $\boldsymbol{x}=\sqrt{-\mathbf{8 1}}$ which is not a real number.
4. Solve $\frac{\frac{\boldsymbol{x}}{\mathbf{6}}}{\mathbf{9}}=\frac{\mathbf{9}}{\frac{\boldsymbol{x}}{\mathbf{6}}}$ by using the square root property.

## Solution:

$$
\begin{aligned}
\frac{x}{\frac{x}{9}} & =\frac{9}{\frac{x}{6}} \\
\left(\frac{x}{6}\right)^{2} & =9^{2} \\
\frac{x}{6} & = \pm 9 \\
x & = \pm 54
\end{aligned}
$$

5. Solve $(\boldsymbol{x}-\mathbf{1 2})^{\mathbf{2}}=\mathbf{1 0 0}$ by using the square root property.

Solution:

$$
\begin{aligned}
(x-12)^{2} & =100 \\
\sqrt{(x-12)^{2}} & = \pm \sqrt{100} \\
x-12 & = \pm 10 \\
x & =12 \pm 10
\end{aligned}
$$

Thus $x=12+10=22$ or $x=12-10=2$.
6. Solve $\mathbf{6 4}(\boldsymbol{x}-\mathbf{5})^{\mathbf{2}}-\mathbf{2 5}=\mathbf{0}$ by using the square root property.

## Solution:

$$
\begin{aligned}
64(x-5)^{2}-25 & =0 \\
64(x-5)^{2} & =25 \\
\sqrt{64(x-5)^{2}} & = \pm \sqrt{25} \\
8(x-5) & = \pm 5 \\
x-5 & = \pm \frac{5}{8} \\
x & =5 \pm \frac{5}{8}
\end{aligned}
$$

Thus $x=\frac{40}{8}+\frac{5}{8}=\frac{45}{8}$
or
$x=\frac{40}{8}-\frac{5}{8}=\frac{35}{8}$.

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## Chapter 68

## Solving Equations by Completing the Square

68.1 Youtube
https://www.youtube.com/playlist?list=PL26AB4293717219C2\&feature=view_all
68.2 Basics


### 68.3 Examples

## Example 1:

Find $a$ and $b$ to complete the square in $(x+a)^{2}=x^{2}+\mathbf{1 2 x}+\boldsymbol{b}$.
Solution:

$$
(x+\underset{\uparrow}{6})^{2}=x^{2}+12 x+36
$$

$a=6$ and $b=36$.
Example 2:
Find $a$ and $b$ to complete the square in $(x-a)^{2}=x^{2}-20 x+b$.

## Solution:

$$
(x-10)^{2}=x^{2}-20 x+100
$$

$a=10$ and $b=100$.
Example 3:
Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(\boldsymbol{x}+\boldsymbol{a})^{2}=\boldsymbol{x}^{2}+\boldsymbol{5} \boldsymbol{x}+\boldsymbol{b}$.

## Solution:

$$
\left(x+\frac{5}{2}\right)^{2}=x^{2}+5 x+\frac{25}{4}
$$

$a=\frac{5}{2}$ and $b=\frac{25}{4}$.
Example 4:
Find $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ to complete the square in $\mathbf{3} \boldsymbol{x}^{2}+\boldsymbol{5} \boldsymbol{x}$ to get $\boldsymbol{a}(\boldsymbol{x}+\boldsymbol{b})^{2}+\boldsymbol{c}$.

## Solution:

$$
\begin{array}{rlr} 
& 3 x^{2}+5 x & \\
= & 3\left(x^{2}+\frac{5}{3} x\right) & \text { coefficient of } x \text { must be } 1 \\
= & 3\left[x^{2}+\frac{5}{3} x+\left(\frac{5}{6}\right)^{2}-\left(\frac{5}{6}\right)^{2}\right] & \text { need } a^{2}=\left(\frac{5}{6}\right)^{2}, \text { but can only add } 0 \\
= & 3\left[\left(x+\frac{5}{6}\right)^{2}-\left(\frac{5}{6}\right)^{2}\right] & \text { First } 3 \text { terms: perfect square } \\
= & 3\left(x+\frac{5}{6}\right)^{2}-3\left(\frac{\mathbf{2 5}}{36}\right) & \\
= & 3\left(x+\frac{5}{6}\right)^{2}-\left(\frac{\mathbf{2 5}}{12}\right) &
\end{array}
$$

$a=3, b=\frac{5}{6}$, and $c=-\frac{25}{12}$.
Example 5:
Solve $\boldsymbol{x}^{2}+\mathbf{1 6 x}+\mathbf{6 0}=\mathbf{0}$ by completing the square.

## Solution:

$$
\begin{aligned}
x^{2}+16 x+60 & =0 & & \\
x^{2}+16 x & =-\mathbf{6 0} & & \text { Complete the square. Use } 2 \text { left terms. } \\
x^{2}+16 x+64 & =\mathbf{6 4 - 6 0} & & \text { Square half of } 16 . \text { Add to both sides. } \\
(x+8)^{2} & =4 & & \text { perfect square } \\
x+8 & = \pm \mathbf{2} & & \text { square root property } \\
x & =-8 \pm 2 & & \text { subtract } 8 \text { from both sides. }
\end{aligned}
$$

Thus $x=-8+2=-6$ and $x=-8-2=-10$
Example 6:
Solve $x^{2}+16 x+50=0$ by completing the square.

## Solution:

$$
\begin{array}{rlrl}
x^{2}+16 x+50 & =0 & & \\
x^{2}+16 x & =-50 & & \text { Complete the square. Use } 2 \text { left terms. } \\
x^{2}+16 x+64 & =\mathbf{6 4 - 5 0} & & \text { Square half of } 16 . \text { Add } \mathbf{6 4} \text { to both sides. } \\
(x+8)^{2} & =\mathbf{1 4} & & \text { First three terms: perfect square. } \\
x+8 & = \pm \sqrt{\mathbf{1 4}} & & \text { square root property } \\
x & =-8 \pm \sqrt{\mathbf{1 4}} &
\end{array}
$$

Thus $x=-8+\sqrt{14}$ and $x=-8-\sqrt{14}$
Example 7:
Solve $3 x^{2}+5 x-7=0$ by completing the square.

## Solution:

$$
\begin{array}{rlrl}
3 x^{2}+5 x-7 & =0 & & \\
3 x^{2}+5 x & =\mathbf{7} & & \text { "Move" the constant out of the way } \\
x^{2}+\frac{5}{3} x & =\frac{7}{3} & & \text { Divide by the coefficient of } x^{2} \\
x^{2}+\frac{5}{3} x+\left(\frac{5}{6}\right)^{2} & =\left(\frac{5}{6}\right)^{2}+\frac{7}{3} & & \text { Add square of } \frac{1}{2} \text { coefficient } \\
\left(x+\frac{5}{6}\right)^{2} & =\frac{\mathbf{2 5}}{36}+\frac{7}{3} \cdot \frac{12}{12} & & \text { of } x \text { to both sides. } \\
& &
\end{array}
$$

$$
\begin{array}{rlrl}
\left(x+\frac{5}{6}\right)^{2} & =\frac{25}{36}+\frac{84}{36} & & \text { Rewrite fractions. } \\
\left(x+\frac{5}{6}\right)^{2} & =\frac{109}{36} & & \text { Add fractions. } \\
x+\frac{5}{6} & = \pm \sqrt{\frac{109}{36}} & & \text { Take } \sqrt{ } \text { of both sides. Don't forget } \pm . \\
x & =-\frac{5}{6} \pm \frac{\sqrt{109}}{6} & \text { Isolate } x . \\
x & =\frac{-5 \pm \sqrt{109}}{6} & \text { Answer could be (and is here) irrational. }
\end{array}
$$

If you understand these steps you should have no difficulty developing the famous quadratic formula.

### 68.4 Exercise 68

1. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(x+a)^{2}=x^{2}+\mathbf{5 0 x}+\boldsymbol{b}$.
2. Find $a$ and $b$ to complete the square in $(x-a)^{2}=x^{2}-\mathbf{3 0 x}+\boldsymbol{b}$.
3. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(x+a)^{2}=x^{2}+7 x+b$.
4. Find $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ to complete the square in $\mathbf{3} \boldsymbol{x}^{2}+5 \boldsymbol{x}$ to get $\boldsymbol{a}(\boldsymbol{x}+\boldsymbol{b})^{2}+\boldsymbol{c}$.
5. Solve $\boldsymbol{x}^{2}-\mathbf{1 2 x}+\mathbf{3 5}=\mathbf{0}$ by completing the square.
6. Solve $\boldsymbol{x}^{2}+\mathbf{2 4 x}+\mathbf{4 0}=\mathbf{0}$ by completing the square.
7. Solve $\mathbf{5} \boldsymbol{x}^{2}+\mathbf{7 x}-\mathbf{9}=\mathbf{0}$ by completing the square.

## STOP!

1. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(x+a)^{2}=x^{2}+50 x+b$.

## Solution:

$$
(x+25)^{2}=x^{2}+50 x+625
$$

$a=25$ and $b=625$.
2. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(x-a)^{2}=x^{2}-\mathbf{3 0 x}+\boldsymbol{b}$.

## Solution:


$a=15$ and $b=225$.
3. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ to complete the square in $(x+\boldsymbol{a})^{2}=x^{2}+7 x+b$.

Solution:
$\left(x+\frac{7}{2}\right)^{2}=x^{2}+7 x+\frac{49}{4}$
$a=\frac{7}{2}$ and $b=\frac{49}{4}$.
4. Find $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ to complete the square in $\mathbf{3} \boldsymbol{x}^{2}+\mathbf{5} \boldsymbol{x}$ to get $\boldsymbol{a}(\boldsymbol{x}+\boldsymbol{b})^{2}+\boldsymbol{c}$.

## Solution:

$$
\begin{aligned}
& 5 x^{2}+9 x \\
= & 5\left(x^{2}+\frac{9}{5} x\right) \\
= & 5\left[x^{2}+\frac{9}{5} x+\left(\frac{9}{10}\right)^{2}-\left(\frac{9}{10}\right)^{2}\right] \\
= & 5\left(x+\frac{9}{10}\right)^{2}-5\left(\frac{81}{100}\right) \\
= & 5\left(x+\frac{9}{10}\right)^{2}-\frac{81}{20} \\
a= & 5, b=\frac{9}{10}, \text { and } c=-\frac{81}{20}
\end{aligned}
$$

5. Solve $\boldsymbol{x}^{\mathbf{2}}-\mathbf{1 2 x}+\mathbf{3 5}=\mathbf{0}$ by completing the square.

## Solution:

$$
\begin{aligned}
x^{2}-12 x+35 & =0 & & \\
x^{2}-12 x & =-35 & & \text { Complete square. Use } 2 \text { left terms. } \\
x^{2}-12 x+36 & =-35+36 & & \text { Square } \frac{1}{2} \text { of 12. Add to } 2 \text { sides. } \\
(x-6)^{2} & =1 & & \\
x-6 & = \pm 1 & & \text { Square root property } \\
x & =6 \pm 1 & &
\end{aligned}
$$

Thus $x=6+1=7$ and $x=6-1=5$
6. Solve $\boldsymbol{x}^{\mathbf{2}}+\mathbf{2 4 x}+\mathbf{4 0}=\mathbf{0}$ by completing the square.

## Solution:

$$
\begin{aligned}
x^{2}+24 x+40 & =0 & & \\
x^{2}+24 x & =-40 & & \text { Complete square. Use } 2 \text { left terms. } \\
x^{2}+24 x+144 & =144-40 & & \text { Square } \frac{1}{2} \text { of } 24 . \text { Add to } 2 \text { sides. } \\
(x+12)^{2} & =104 & & \\
x+12 & = \pm \sqrt{104} & & \text { square root property } \\
x & =-12 \pm \sqrt{104} & &
\end{aligned}
$$

Thus $x=-12+\sqrt{104}$ and $x=-12-\sqrt{104}$
7. Solve $\mathbf{5} \boldsymbol{x}^{2}+7 \boldsymbol{x}-\mathbf{9}=\mathbf{0}$ by completing the square.

## Solution:

$$
\begin{array}{rlrl}
5 x^{2}+7 x-9 & =0 & \\
5 x^{2}+7 x & =9 & & \text { "Move" the constant out of the way } \\
x^{2}+\frac{7}{5} x & =\frac{9}{5} & & \text { Divide by the coefficient of } x^{2} \\
x^{2}+\frac{7}{5} x+\left(\frac{7}{10}\right)^{2} & =\left(\frac{7}{10}\right)^{2}+\frac{9}{5} & \text { Square } \frac{1}{2} \text { coefficient of } x \text {. Add to } 2 \text { sides. }
\end{array}
$$

$$
\left(x+\frac{7}{10}\right)^{2}=\frac{49}{100}+\frac{9}{5} \cdot \frac{20}{20} \quad \text { Square is completed. Use LCD. }
$$

$$
\left(x+\frac{7}{10}\right)^{2}=\frac{49}{100}+\frac{180}{100} \quad \text { Rewrite fractions. }
$$

$$
\left(x+\frac{7}{10}\right)^{2}=\frac{229}{100} \quad \text { Add fractions }
$$

$$
x+\frac{7}{10}= \pm \sqrt{\frac{229}{100}} \quad \sqrt{ } \text { of both sides. Don't forget } \pm
$$

$$
x=-\frac{7}{10} \pm \frac{\sqrt{229}}{10} \quad \text { Isolate } x
$$

$x=\frac{-7 \pm \sqrt{\mathbf{2 2 9}}}{10} \quad$ Answer could be (is here) irrational.

## Chapter 69

## The Quadratic Formula

### 69.1 Youtube

https://www.youtube.com/playlist?list=PL8745BA34AFA6CF79\&feature=view_all

### 69.2 Basics

Solve $\mathbf{3} \boldsymbol{x}^{2}+\mathbf{5 x}-\mathbf{7}=\mathbf{0}$ and $\boldsymbol{a} \boldsymbol{x}^{\mathbf{2}}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}$ by completing the square.

$$
\begin{array}{rl}
3 x^{2}+5 x-7 & =0 \\
3 x^{2}+5 x & =7 \\
x^{2}+\frac{5}{3} x & =\frac{7}{3} \\
x^{2}+\frac{5}{3} x+\left(\frac{5}{6}\right)^{2} & =\left(\frac{5}{6}\right)^{2}+\frac{7}{3} \\
\left(x+\frac{5}{6}\right)^{2} & =\frac{25}{36}+\frac{7}{3} \cdot \frac{12}{12} \left\lvert\, \begin{aligned}
a x^{2}+b x+c & =0 \\
a x^{2}+b x & =-c \\
x^{2}+\frac{b}{a} x & =\frac{-c}{a} \\
x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2} & =\left(\frac{b}{2 a}\right)^{2}+\frac{-c}{a} \\
\left(x+\frac{5}{6}\right)^{2} & =\frac{25}{36}+\frac{84}{36} \\
\left(x+\frac{b}{2 a}\right)^{2} & =\frac{b^{2}}{4 a^{2}}+\frac{-c}{a} \cdot \frac{4 a}{4 a} \\
\left(x+\frac{5}{6}\right)^{2} & =\frac{109}{36} \\
x+\frac{5}{6} & =\frac{ \pm \sqrt{\frac{109}{36}}}{\left(x+\frac{b}{2 a}\right)^{2}} \\
x & =\frac{b^{2}}{4 a^{2}}+\frac{-4 a c}{4 a^{2}} \\
x & =-\frac{5}{6} \pm \frac{\sqrt{109}}{6} \\
\left(x+\frac{b}{2 a}\right)^{2} & =\frac{b^{2}-4 a c}{4 a^{2}} \\
x+\frac{b}{a} & = \pm \sqrt{\frac{b^{2}-4 a c}{4 a^{2}}} \\
x & =\frac{-5 \pm \sqrt{109}}{6}
\end{aligned} \quad x=\frac{b}{a} \pm \frac{\sqrt{b^{2}-4 a c}}{2 a}\right. \\
x & x
\end{array}
$$

The quadratic formula is very important. Memorize it. It is so important that if you reach age $\mathbf{9 0}$ and you forgot everything else, you should still remember the quadratic formula. When you reach age $\mathbf{9 0}$, I will come by to check.

## $a x^{2}+b x+c=0$

$\Rightarrow$


We can develop a special formula if the coefficient of $\boldsymbol{x}$, namely $\boldsymbol{b}$, is even. $\boldsymbol{b}=\mathbf{2} \overline{\boldsymbol{b}}$.
Given $\boldsymbol{a} \boldsymbol{x}^{2}+\mathbf{2} \overline{\boldsymbol{b}} \boldsymbol{x}+\boldsymbol{c}=\mathbf{0}$ then

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-2 \bar{b} \pm \sqrt{(2 \bar{b})^{2}-4 a c}}{2 a} \\
& =\frac{-2 \bar{b} \pm \sqrt{4 \bar{b}^{2}-4 a c}}{2 a} \\
& =\frac{-2 \bar{b} \pm \sqrt{4\left(\bar{b}^{2}-a c\right)}}{2 a} \\
& =\frac{-2 \bar{b} \pm 2 \sqrt{\bar{b}^{2}-a c}}{2 a} \\
& =\frac{2\left(-\bar{b} \pm \sqrt{\bar{b}^{2}-a c}\right)}{2 a} \\
& =\frac{-\bar{b} \pm \sqrt{\bar{b}^{2}-a c}}{a}
\end{aligned}
$$

The expression $\boldsymbol{D}=\boldsymbol{b}^{2}-\mathbf{4 a c}$ under the square root of the general quadratic formula is called the discriminant. It can be used to determine the nature of the roots (solutions) of the quadratic equation.

If $\boldsymbol{D}>\boldsymbol{0}$ the quadratic equation has two distinct real roots. If in addition $\boldsymbol{D}$ is a perfect square, then the real distinct roots are rational.

If $\boldsymbol{D}=\mathbf{0}$ the solution is a double (real) root (one answer).

If $\boldsymbol{D}<\mathbf{0}$ the solutions are not real.

### 69.3 Examples

Example 1:
Solve $\boldsymbol{x}^{2}+\mathbf{1 6 x}+\mathbf{6 0}=\mathbf{0}$ by the quadratic formula.
Solution:
$a=1, b=16$, and $c=60$.
$D=b^{2}-4 a c=16^{2}-4(1)(60)=256-240>0$. We will get two real distinct roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-16 \pm \sqrt{16^{2}-4(1)(60)}}{2(1)} \\
& =\frac{-16 \pm \sqrt{256-240}}{2} \\
& =\frac{-16 \pm \sqrt{16}}{2} \\
& =\frac{-16 \pm 4}{2} \\
& =\frac{2(-8 \pm 2)}{2} \\
& =-8 \pm 2
\end{aligned}
$$

So $\boldsymbol{x}=\mathbf{- 1 0}$ or $\mathbf{- 6}$.
Since $b=16=\mathbf{2 ( 8 )}$ we could have used our special formula with $\bar{b}=\mathbf{8}$.

$$
\begin{aligned}
x & =\frac{-\bar{b} \pm \sqrt{(\bar{b})^{2}-a c}}{a} \\
& =\frac{-8 \pm \sqrt{8^{2}-(1)(60)}}{(1)} \\
& =-8 \pm \sqrt{64-60} \\
& =-8 \pm \sqrt{4} \\
& =-8 \pm 2
\end{aligned}
$$

So $\boldsymbol{x}=\mathbf{- 1 0}$ or $\mathbf{- 6}$.
Example 2:
Solve $\boldsymbol{x}^{2}-5 x-20=0$ by using the quadratic formula.
Solution:
$x^{2}-5 x-20=0$
$a=1, b=-5$, and $c=-20$.
$D=b^{2}-4 a c=(-5)^{2}-4(1)(-20)=25+20>0$. We will get two real distinct roots.
$x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$
$=\frac{-(-5) \pm \sqrt{(-5)^{2}-4(1)(-20)}}{2(1)}$
$=\frac{5 \pm \sqrt{25+80}}{2}$
$=\frac{5 \pm \sqrt{105}}{2}$

Example 3:
Solve $2 x^{2}+7 x=5$ by using the quadratic formula.
Solution:

$$
\begin{array}{r}
2 x^{2}+7 x=5 \\
2 x^{2}+7 x-5=0
\end{array}
$$

$a=2, b=7$, and $c=-5$.
$D=b^{2}-4 a c=7^{2}-4(2)(-5)=49+40>0$. We will get two real distinct roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-7 \pm \sqrt{(7)^{2}-4(2)(-5)}}{2(2)} \\
& =\frac{-7 \pm \sqrt{49+40}}{4} \\
& =\frac{-7 \pm \sqrt{89}}{4}
\end{aligned}
$$

Example 4:

Solve $\mathbf{3} \boldsymbol{x}^{2}=10 x-8$ by using the quadratic formula.

## Solution:

$$
\begin{aligned}
3 x^{2} & =10 x-8 \\
3 x^{2}-10 x+8 & =0
\end{aligned}
$$

$a=3, b=-10$, and $c=8$.
$D=b^{2}-4 a c=10^{2}-4(3)(8)=100-96>0$. We will get two real distinct roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(-10) \pm \sqrt{(10)^{2}-4(3)(8)}}{2(3)} \\
& =\frac{10 \pm \sqrt{100-96}}{6} \\
& =\frac{10 \pm \sqrt{4}}{6} \\
& =\frac{10 \pm 2}{6} \\
& =\frac{2(5 \pm 1)}{6}=\frac{5 \pm 1}{3} \\
x= & \frac{5+1}{3}=\frac{6}{3}=2 \text { or } x=\frac{5-1}{3}=\frac{4}{3}
\end{aligned}
$$

Note: $\boldsymbol{b}=\mathbf{- 1 0}=\mathbf{2}(\mathbf{- 5})=\overline{\boldsymbol{b}}$. We can use the special formula

$$
\begin{aligned}
x & =\frac{-\bar{b} \pm \sqrt{\bar{b}^{2}-a c}}{a} \\
& =\frac{-(-5) \pm \sqrt{(5)^{2}-(3)(8)}}{3} \\
& =\frac{5 \pm \sqrt{25-24}}{3} \\
& =\frac{5 \pm 1}{3}
\end{aligned}
$$

Example 5:
Solve $3 x^{2}=10 x-9$ by using the quadratic formula.

## Solution:

$$
\begin{aligned}
& \qquad \begin{array}{l}
3 x^{2}=10 x-9 \\
3 x^{2}-10 x+9=0 \\
a=3, b=-10, \text { and } c=9 . \text { (Redo with } \bar{b}=-5) \\
D= \\
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
x=\frac{-(-10) \pm \sqrt{(10)^{2}-4(3)(9)}}{2(3)} \\
= \\
=\frac{10 \pm \sqrt{100-108}}{6} \\
=
\end{array} \\
& =\frac{10 \pm \sqrt{-8}}{6}
\end{aligned}
$$

This problem does not have real solutions.
Example 6:
Solve $9 x^{2}+30 x+25$ by using the quadratic formula.

## Solution:

$$
9 x^{2}+30 x+25=0
$$

$a=9, b=30$, and $\boldsymbol{c}=25$. (Redo with $\bar{b}=15$ )
$D=b^{2}-4 a c=30^{2}-4(9)(25)=900-900=0$. We will get one (double) root.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-30 \pm \sqrt{(30)^{2}-4(9)(25)}}{2(9)} \\
& =\frac{-30 \pm \sqrt{900-900}}{18} \\
& =\frac{-30}{18}=\frac{15}{9}
\end{aligned}
$$

This problem has a double root.

### 69.4 Exercise 69

1. Solve $\boldsymbol{x}^{2}+\mathbf{1 2 x}+\mathbf{3 5}=\mathbf{0}$ by the quadratic formula.
2. Solve $\boldsymbol{x}^{\mathbf{2}}-\mathbf{7 x}-\mathbf{1 0}=\mathbf{0}$ by using the quadratic formula.
3. Solve $3 x^{2}+8 x=6$ by using the quadratic formula.
4. Solve $\mathbf{5} \boldsymbol{x}^{2}=\mathbf{1 2 x}+\mathbf{2}$ by using the quadratic formula.
5. Solve $2 x^{2}=9 x-8$ by using the quadratic formula.
6. Solve $\mathbf{1 6} \boldsymbol{x}^{2}+\mathbf{2 4 x}+\mathbf{9}$ by using the quadratic formula.

## STOP!

1. Solve $\boldsymbol{x}^{2}+\mathbf{1 2 x}+\mathbf{3 5}=\mathbf{0}$ by the quadratic formula.

## Solution:

$a=1, b=12$, and $c=35$. (Redo with $\bar{b}=6$ )
$D=b^{2}-4 a c=12^{2}-4(1)(35)=144-140=4>0$. We will get two real distinct (rational) roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-12 \pm \sqrt{12^{2}-4(1)(35)}}{2(1)} \\
& =\frac{-12 \pm \sqrt{144-140}}{2} \\
& =\frac{-16 \pm \sqrt{4}}{2} \\
& =\frac{-12 \pm 2}{2} \\
& =\frac{2(-6 \pm 1)}{2}=-6 \pm 1
\end{aligned}
$$

So $\boldsymbol{x}=-\mathbf{7}$ or $\mathbf{- 5}$.
2. Solve $\boldsymbol{x}^{\mathbf{2}}-\mathbf{7 x}-\mathbf{1 0}=\mathbf{0}$ by using the quadratic formula.

## Solution:

$$
x^{2}-7 x-10=0
$$

$a=1, b=-7$, and $c=-10$.
$D=b^{2}-4 a c=(-7)^{2}-4(1)(-10)=49+40>0$. We will get two real distinct roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(-7) \pm \sqrt{(-7)^{2}-4(1)(-10)}}{2(1)} \\
& =\frac{7 \pm \sqrt{49+40}}{2} \\
& =\frac{5 \pm \sqrt{89}}{2}
\end{aligned}
$$

3. Solve $3 x^{2}+8 x=6$ by using the quadratic formula.

## Solution:

$$
\begin{aligned}
& 3 x^{2}+8 x=6 \\
& 3 x^{2}+8 x-6=0 \\
& a= 3, b=8, \text { and } c=-6 .(\text { Redo with } \bar{b}=4) \\
& D=b^{2}-4 a c=8^{2}-4(3)(-6)=64+72>0 . \text { We will get two real distinct roots. } \\
& x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
&=\frac{-8 \pm \sqrt{(8)^{2}-4(3)(-6)}}{2(3)} \\
&=\frac{-8 \pm \sqrt{64+72}}{6} \\
&=\frac{-8 \pm \sqrt{136}}{6} \\
&=\frac{-8 \pm \sqrt{4 \cdot 34}}{6} \\
&=\frac{-8 \pm 2 \sqrt{4}}{6} \\
&=\frac{-4 \pm \sqrt{4}}{3}
\end{aligned}
$$

4. Solve $\mathbf{5} \boldsymbol{x}^{2}=12 x+2$ by using the quadratic formula.

## Solution:

$$
\begin{aligned}
5 x^{2} & =12 x+2 \\
5 x^{2}-12 x-2 & =0
\end{aligned}
$$

$a=5, b=-12$, and $\boldsymbol{c}=-2$. (Redo with $\bar{b}=6$ )
$D=b^{2}-4 a c=12^{2}-4(5)(-2)=144+40>0$. We will get two distinct real roots.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(-12) \pm \sqrt{(12)^{2}-4(5)(-2)}}{2(5)} \\
& =\frac{12 \pm \sqrt{144+40}}{10} \\
& =\frac{12 \pm \sqrt{184}}{10} \\
& =\frac{12 \pm \sqrt{4 \cdot 46}}{10} \\
& =\frac{12 \pm 2 \sqrt{46}}{10} \\
& =\frac{2(6 \pm \sqrt{46})}{2 \cdot 5} \\
& =\frac{6 \pm \sqrt{46}}{5}
\end{aligned}
$$

5. Solve $2 \boldsymbol{x}^{2}=\mathbf{9 x}-\mathbf{8}$ by using the quadratic formula.

Solution:

$$
\begin{aligned}
& \qquad \begin{aligned}
2 x^{2} & =9 x-8 \\
2 x^{2}-9 x+8 & =0
\end{aligned} \\
& \begin{aligned}
a= & 2, b=-9, \text { and } c=8 \\
D= & b^{2}-4 a c=(-9)^{2}-4(2)(8)=81-64>0 . \text { We will get two real distinct solutions. } \\
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
= & \frac{-(-9) \pm \sqrt{(-9)^{2}-4(2)(8)}}{2(2)} \\
= & \frac{9 \pm \sqrt{81-64}}{4} \\
= & \frac{9 \pm \sqrt{17}}{4}
\end{aligned}
\end{aligned}
$$

6. Solve $16 \boldsymbol{x}^{2}+\mathbf{2 4 x}+\mathbf{9}$ by using the quadratic formula.

## Solution:

$16 x^{2}+24 x+9=0$
$a=16, b=24$, and $\boldsymbol{c}=9$. (Redo with $\bar{b}=12$ )
$D=b^{2}-4 a c=24^{2}-4(16)(9)=576-576=0$. We will get one (double) root.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-24 \pm \sqrt{(24)^{2}-4(16)(9)}}{2(16)} \\
& =\frac{-24 \pm \sqrt{579-579}}{32} \\
& =\frac{-24}{32}=-\frac{3}{8}
\end{aligned}
$$

This problem has a double root.

## Chapter 70

## Solving Application Problems Involving Quadratic Equations

### 70.1 Youtube

https://www.youtube.com/playlist?list=PLA2EBB31FBA33D79F\&feature=view_all

### 70.2 Basics

Remember the quadratic formula:

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

### 70.3 Examples

## Example 1:

A small pipe can fill a pool in $\mathbf{4}$ hours more than a larger pipe. Working togethet they fill the pool in $\mathbf{3} \frac{\mathbf{3}}{\mathbf{4}}$ hours. How long will each pipe take working by itself?
Solution:
Let $\boldsymbol{x}$ be the number of hours it takes the large pipe to fill the pool. Then $\boldsymbol{x}+\boldsymbol{4}$ will be the time it takes the small pipe.

|  | Number of hours | Portion of pool filled in one hour. |
| :---: | :---: | :---: |
| Large pipe | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| Small pipe | $\boldsymbol{x}+\mathbf{4}$ | $\frac{\mathbf{1}}{\boldsymbol{x + 4}}$ |
| Both pipes | $\mathbf{3} \frac{\mathbf{3}}{\mathbf{4}}=\frac{\mathbf{1 5}}{\mathbf{4}}$ | $\frac{\mathbf{4}}{\mathbf{1 5}}$ |

The pool portion filled in $\mathbf{1}$ hour by the large pipe plus the portion filled by the small pipe equals the portion filled by both pipes per hour.

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{x+4} & =\frac{4}{15} \\
\frac{15 x(x+4)}{x}+\frac{15 x(x+4)}{x+4} & =\frac{4(15) x(x+4)}{15} \\
15(x+4)+15 x & =4 x(x+4) \\
15 x+60+15 x & =4 x^{2}+16 x \\
30 x+60 & =4 x^{2}+16 x \\
0 & =4 x^{2}+16 x-30 x-60 \\
4 x^{2}-14 x-60 & =0 \\
2 x^{2}-7 x-30 & =0 \\
x & =\frac{-(-7) \pm \sqrt{(-7)^{2}-4(2)(-30)}}{2(2)} \\
x & =\frac{7 \pm \sqrt{49+240}}{4} \\
x & =\frac{7 \pm \sqrt{289}}{4}=\frac{7 \pm 17}{4}
\end{aligned}
$$

Drop the negative answer. A pipe time can't be negative.
So $x=\frac{7+17}{4}=6$.
The large pipe will take $\mathbf{6}$ hours to fill the tank by itself.
The small pipe takes $\mathbf{6}+\mathbf{4}=\mathbf{1 0}$ hours to fill the tank by itself.
Example 2:
A ball is thrown upward at $\mathbf{3 2} \mathrm{ft}$ per sec from ground level. It is to dislodge a sheet in telephone wires 40 ft above the ground. The distance traveled (height of the ball) at time $\boldsymbol{t}$ is $h(t)=-16 t^{2}+v t+6=-16 t^{2}+32 t+6$.
(a) What is the maximum height reached by the ball?
(b) Can the sheet be dislodged by the ball?
(c) If the ball fails to reach the sheet, what is the least velocity $\boldsymbol{v}$ (rounded to one decimal) for the ball to reach the sheet?

## Solution:

$$
\begin{aligned}
h(t) & =-16 t^{2}+32 t+6 \\
& =-16\left(t^{2}-2 t\right)+6 \\
& =-16\left(t^{2}-2 t+1\right)+16+6 \\
& =-16(t-1)^{2}+16+6 \\
& =-16(t-1)^{2}+22
\end{aligned}
$$

(a) The maximum height that can be reached is $\mathbf{2 2} \mathrm{ft}$. That occurs $\mathbf{1}$ second after launch. (Subtract the minimum 0 from 22)
(b) Since the sheet is $\mathbf{4 0} \mathrm{ft}$ above ground, the ball will not dislodge it.
(c) Let's replace $\boldsymbol{h}(\boldsymbol{t})$ by $\mathbf{4 0}$ and $\mathbf{3 2 \boldsymbol { t }}$ by $\boldsymbol{v} \boldsymbol{t}$.

$$
\begin{aligned}
40 & =-16 t^{2}+v t+6 \\
0 & =-16 t^{2}+v t+6-40 \\
-16 t^{2}+v t+6-40 & =0 \\
-16\left(t^{2}+\frac{v}{-16}\right) t+6-40 & =0 \\
-16\left(t^{2}-\frac{v}{16}+\frac{v^{2}}{32^{2}}\right)+\frac{16\left(v^{2}\right)}{32^{2}}-34 & =0 \\
-16\left(t-\frac{v}{32}\right)^{2}+\frac{16\left(v^{2}\right)}{32^{2}}-34 & =0
\end{aligned}
$$

The maximum height is reached when the square term $\boldsymbol{t}-\frac{\boldsymbol{v}}{\mathbf{3 2}}$ disappears (is 0 ). At that moment the height is $\frac{\mathbf{1 6 ( \boldsymbol { v } ^ { 2 } )}}{\mathbf{3 2}^{\mathbf{2}}}-\mathbf{3 4}$ which needs to be 40 at a minimum. Thus

$$
\begin{aligned}
\frac{16\left(v^{2}\right)}{32^{2}}-34 & =40 \\
\frac{16\left(v^{2}\right)}{32^{2}} & =74 \\
v^{2} & =32(2)(74) \\
v & =\sqrt{32(2)(74)} \approx 69
\end{aligned}
$$

If the ball is thrown upward at a minimum speed of $\mathbf{6 9} \mathrm{ft}$ per second, it will reach the sheet.
Example 3:
A motorcycle takes one hour longer to travel 150 miles than a car. The speed of the car is $\mathbf{5}$ miles per hour more than that of the motorcycle. What is the speed of the motorcycle?

## Solution:

Let $\boldsymbol{x}$ be the speedof the car. Then $\boldsymbol{x}+\mathbf{5}$ is the speed of the motorcycle.

|  | Rate (speed) | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Motorcycle | $\boldsymbol{x}$ | $\frac{\mathbf{1 5 0}}{\boldsymbol{x}}$ | $\mathbf{1 5 0}$ |
| Car | $\boldsymbol{x}+\mathbf{5}$ | $\frac{\mathbf{1 5 0}}{\boldsymbol{x + 5}}$ | $\mathbf{1 5 0}$ |

The motorcycle takes 1 hour longer to travel 150 miles than the car.

$$
\begin{gathered}
\frac{150}{x}=\frac{150}{x+5}+1 \\
\frac{150 x(x+5)}{x}=\frac{150 x(x+5)}{x+5}+x(x+5) \\
150(x+5)=150 x+x^{2}+5 x \\
150 x+5(150)=150 x+x^{2}+5 x \\
750=x^{2}+5 x \\
x^{2}+5 x-750=0 \\
x=\frac{-5 \pm \sqrt{5^{2}-4(1)(-750)}}{2(1)} \\
=\frac{-5 \pm \sqrt{5^{2}+3,000}}{2} \\
=\frac{-5 \pm \sqrt{3,025}}{2} \\
=\frac{-5 \pm 55}{2}
\end{gathered}
$$

Reject the negative answer because rates are not negative (in our context).
$x=\frac{-5+55}{2}=25$
The speed of the motorcycle is 25 mph .

### 70.4 Exercise 70

1. A small pipe can fill a pool in 2 hours more than a larger pipe. Working together they fill the pool in $4 \frac{19}{20}$ hours. How long will each pipe take working by itself?
2. A ball is thrown upward at $\mathbf{6 4} \mathbf{~ m p h}$ from ground level. It is to dislodge a sheet in telephone wires $\mathbf{7 5}$ ft above the height above ground. The distance traveled (height of the ball) at time $\boldsymbol{t}$ is $h(t)=-16 t^{2}+v t+6=-16 t^{2}+64 t+6$.
(a) What is the maximum height reached by the ball?
(b) Can the sheet be dislodged by the ball?
(c) If the ball does not go as high as the sheet, what is the minimum velocity $\boldsymbol{v}$ (rounded to one decimal if needed) for the ball to reach the sheet?
3. A motorcycle takes one hour longer to travel $\mathbf{1 8 0}$ miles than a car. The speed of the car is $\mathbf{6}$ miles per hour more than that of the motorcycle. What is the speed of the motorcycle?

## STOP!

1. A small pipe can fill a pool in $\mathbf{2}$ hours more than a larger pipe. Working together they fill the pool in $4 \frac{19}{20}$ hours. How long will each pipe take working by itself?

## Solution:

Let $\boldsymbol{x}$ be the number of hours it takes the large pipe to fill the pool. Then $\boldsymbol{x}+\mathbf{2}$ will be the time it takes the small pipe.

|  | Number of hours | Portion of pool filled in one hour. |
| :---: | :---: | :---: |
| Large pipe | $\boldsymbol{x}$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ |
| Small pipe | $\boldsymbol{x}+\mathbf{2}$ | $\frac{\mathbf{1}}{\boldsymbol{x + 2}}$ |
| Both pipes | $4 \frac{\mathbf{1 9}}{\mathbf{2 0}}=\frac{\mathbf{9 9}}{\mathbf{2 0}}$ | $\frac{\mathbf{2 0}}{\mathbf{9 9}}$ |

The portion of the pool filled in one hour by the large pipe added to the portion of the pipe by the small pipe is the same as the portion filled by both pipes in one hour.

$$
\begin{aligned}
\frac{1}{x}+\frac{1}{x+2} & =\frac{20}{99} \\
\frac{99 x(x+2)}{x}+\frac{99 x(x+2)}{x+2} & =\frac{20(99) x(x+2)}{99} \\
99(x+2)+99 x & =20 x(x+2) \\
99 x+198+99 x & =20 x^{2}+40 x \\
198 x+198 & =20 x^{2}+40 x \\
0 & =20 x^{2}-158 x-198 \\
20 x^{2}-158 x-198 & =0 \\
10 x^{2}-79 x-99 & =0 \\
x & =\frac{-(-79) \pm \sqrt{(-79)^{2}-4(10)(-99)}}{2(10)} \\
& =\frac{79 \pm \sqrt{6241+3960}}{20} \\
& =\frac{79 \pm \sqrt{10201}}{20} \\
& =\frac{79 \pm 101}{5}
\end{aligned}
$$

Drop the negative answer because a pipe cannot fill anything in a negative number of hours.
So $x=\frac{79+101}{20}=\frac{180}{20}=9$.
The large pipe will take $\mathbf{9}$ hours to fill the tank by itself.
The small pipe will take $\mathbf{9 + 2}=\mathbf{1 1}$ hours to fill the tank by itself.
2. A ball is thrown upward at $\mathbf{6 4} \mathrm{ft}$ per second from ground level. It is to dislodge a sheet in telephone wires 75 ft above the height above ground. The distance traveled (height of the ball) at time $\boldsymbol{t}$ is $h(t)=-16 t^{2}+v t+6=-16 t^{2}+64 t+6$.
(a) What is the maximum height reached by the ball?
(b) Can the sheet be dislodged by the ball?
(c) If the ball does not go as high as the sheet, what is the minimum velocity $\boldsymbol{v}$ (rounded to one decimal if needed) for the ball to reach the sheet?

## Solution:

$$
\begin{aligned}
h(t) & =-16 t^{2}+64 t+6 \\
& =-16\left(t^{2}-4 t\right)+6 \\
& =-16\left(t^{2}-4 t+4\right)+64+6 \\
& =-16(t-2)^{2}+70
\end{aligned}
$$

(a) The maximum height that can be reached is $\mathbf{7 0} \mathrm{ft}$. That occurs $\mathbf{2}$ seconds after launch.
(b) Since the sheet is $\mathbf{7 5} \mathrm{ft}$ above ground, the ball will not dislodge it.
(c) Let's replace $\boldsymbol{h}(\boldsymbol{t})$ by $\mathbf{7 5}$ and $\mathbf{6 4 t}$ by $\boldsymbol{v} \boldsymbol{t}$.

$$
\begin{aligned}
75 & =-16 t^{2}+v t+6 \\
0 & =-16 t^{2}+v t+6-75 \\
-16 t^{2}+v t+6-75 & =0 \\
-16\left(t^{2}-\frac{v}{16}+\frac{v^{2}}{32^{2}}\right)+\frac{16\left(v^{2}\right)}{32^{2}}-69 & =0 \\
-16\left(t-\frac{v}{32}\right)^{2}+\frac{16\left(v^{2}\right)}{32^{2}}-69 & =0
\end{aligned}
$$

The maximum height is reached when the square term
$\boldsymbol{t}-\frac{\boldsymbol{v}}{\mathbf{3 2}}$ disappears (is $\mathbf{0}$ ). At that moment the height is
$\frac{16\left(v^{2}\right)}{32^{2}}-69$ which needs to be 75 at a minimum. Thus

$$
\begin{aligned}
\frac{16\left(v^{2}\right)}{32^{2}}-69 & =75 \\
\frac{16\left(v^{2}\right)}{32^{2}} & =144 \\
v^{2} & =32(2)(144) \\
v & =\sqrt{32(2)(144)}=96
\end{aligned}
$$

If the ball is thrown upward at a minimum speed of $\mathbf{9 6 f t}$ per second, it will reach the sheet.
3. A motorcycle takes one hour longer to travel 180 miles than a car. The speed of the car is $\mathbf{6}$ miles per hour more than that of the motorcycle. What is the speed of the motorcycle?

## Solution:

Let $\boldsymbol{x}$ be the speed of the car. Then $\boldsymbol{x}+\mathbf{6}$ is the speed of the motorcycle.

|  | Rate (speed) | $\cdot$ Time | $=$ Distance |
| :---: | :---: | :---: | :---: |
| Motorcycle | $\boldsymbol{x}$ | $\frac{\mathbf{1 8 0}}{\boldsymbol{x}}$ | $\mathbf{1 8 0}$ |
| Car | $\boldsymbol{x}+\mathbf{6}$ | $\frac{\mathbf{1 8 0}}{\boldsymbol{x + 6}}$ | $\mathbf{1 8 0}$ |

A motorcycle takes one hour longer to travel 180 miles than a car.

$$
\begin{aligned}
\frac{180}{x} & =\frac{180}{x+6}+1 \\
\frac{180 x(x+6)}{x} & =\frac{180 x(x+6)}{x+6}+x(x+6) \\
180(x+6) & =180 x+x^{2}+6 x \\
180 x+6(180) & =180 x+x^{2}+6 x \\
1080 & =x^{2}+6 x \\
x^{2}+6 x-1,080 & =0 \quad \text { Use } \bar{b}=3 \\
x & =\frac{-3 \pm \sqrt{3^{2}-(1)(-1,080)}}{(1)} \\
& =-3 \pm \sqrt{9+1,080} \\
& =-3 \pm \sqrt{1,089} \\
& =-3 \pm 33
\end{aligned}
$$

Reject the negative answer because the rate is not negative.
$x=-3+33=30$
The speed of the motorcycle is $\mathbf{3 0} \mathrm{mph}$.

## Chapter 71

## Solving Formulas Involving Quadratics

### 71.1 Youtube

### 71.2 Basics

If $a x^{2}+b x+c=0$ then $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$.

### 71.3 Examples

Example 1:
Solve $\boldsymbol{V}=\boldsymbol{\pi} \boldsymbol{R}^{2} \boldsymbol{H}$ for $\boldsymbol{R}$.
Solution:

$$
\begin{aligned}
V & =\pi R^{2} H \\
R^{2} & =\frac{V}{\pi H} \\
R & = \pm \sqrt{\frac{V}{\pi H}}
\end{aligned}
$$

If $\boldsymbol{V}=\boldsymbol{\pi} \boldsymbol{R}^{\mathbf{2}} \boldsymbol{H}$ is the formula for the volume of a cylinder, then the negative value of $\boldsymbol{R}$ is rejected.
Example 2:
Solve $\boldsymbol{F}=\frac{G m_{1} m_{2}}{d^{2}}$ for $\boldsymbol{d}$.

## Solution:

$$
\begin{aligned}
F & =\frac{G m_{1} m_{2}}{d^{2}} \\
d^{2} & =\frac{G m_{1} m_{2}}{F} \\
d & = \pm \sqrt{\frac{G m_{1} m_{2}}{F}}
\end{aligned}
$$

If $\boldsymbol{F}$ is the force of attraction between two masses, then the negative answer is rejected.
Example 3:
Solve $m=\frac{m_{0}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}$ for $v$.

## Solution:

$$
\begin{aligned}
m & =\frac{m_{0}}{\sqrt{1-\frac{v^{2}}{c^{2}}}} \\
m^{2} & =\frac{m_{0}^{2}}{1-\frac{v^{2}}{c^{2}}} \\
1-\frac{v^{2}}{c^{2}} & =\frac{m_{0}^{2}}{m^{2}} \\
c^{2}-\frac{c^{2} v^{2}}{c^{2}} & =\frac{c^{2} m_{0}^{2}}{m^{2}} \\
c^{2}-v^{2} & =\frac{c^{2} m_{0}^{2}}{m^{2}} \\
c^{2}-\frac{c^{2} m_{0}^{2}}{m^{2}} & =v^{2} \\
v^{2} & =c^{2}-\frac{c^{2} m_{0}^{2}}{m^{2}} \\
v & = \pm \sqrt{c^{2}-\frac{c^{2} m_{0}^{2}}{m^{2}}} \\
& = \pm \sqrt{\frac{c^{2} m^{2}}{m^{2}}-\frac{c^{2} m_{0}^{2}}{m^{2}}}= \pm \sqrt{\frac{c^{2}}{m^{2}}\left(m^{2}-m_{0}^{2}\right)}= \pm \frac{c}{m} \sqrt{m^{2}-m_{0}^{2}}
\end{aligned}
$$

Example 4:
Solve $a^{2}+b^{2}=c^{2}$ for $b$.
Solution:

$$
\begin{aligned}
a^{2}+b^{2} & =c^{2} \\
b^{2} & =c^{2}-a^{2} \\
b & = \pm \sqrt{c^{2}-a^{2}}
\end{aligned}
$$

Example 5:
Solve $m_{0}^{2} c^{4}+\boldsymbol{p}^{2} c^{2}-E^{2}=0$ for $\boldsymbol{c}$.
Solution:

$$
\begin{aligned}
m_{0}^{2} c^{4}+p^{2} c^{2}-E^{2} & =0 \\
m_{0}^{2}\left(c^{2}\right)^{2}+p^{2}\left(c^{2}\right)-E^{2} & =0 \quad \text { Replace } c^{2} \text { by } x \\
m_{0}^{2} x^{2}+p^{2} x-E^{2} & =0 \text { This is the quadratic formula } \\
a x^{2}+b x+c=0 & \text { with } a=m_{0}^{2}, b=p^{2}, c=-E^{2} \\
x & =\frac{-p^{2} \pm \sqrt{\left(p^{2}\right)^{2}-4\left(m_{0}^{2}\right)\left(-E^{2}\right)}}{2 m_{0}^{2}} \\
c^{2} & =\frac{-p^{2} \pm \sqrt{p^{4}+4 m_{0}^{2} E^{2}}}{2 m_{0}^{2}} \\
c & = \pm \sqrt{\frac{-p^{2} \pm \sqrt{p^{4}+4 m_{0}^{2} E^{2}}}{2 m_{0}^{2}}}
\end{aligned}
$$

### 71.4 Exercise 71

1. Solve $\boldsymbol{A}=\pi \boldsymbol{R}^{2}$ for $\boldsymbol{R}$.
2. Solve $\boldsymbol{E}=\boldsymbol{m} \boldsymbol{c}^{\mathbf{2}}$ for $\boldsymbol{c}$.
3. Solve $\boldsymbol{S}=\mathbf{2 \pi} \boldsymbol{R}^{2}+\mathbf{2 \pi R H}$ for $\boldsymbol{R}$.
4. Solve $(\boldsymbol{x}-\boldsymbol{H})^{2}+(\boldsymbol{y}-\boldsymbol{k})^{2}=\boldsymbol{R}^{2}$ for $\boldsymbol{y}$.
5. Solve $\boldsymbol{A}=\boldsymbol{b} \boldsymbol{x}^{2} \boldsymbol{c}$ for $\boldsymbol{x}$.
6. Solve $\boldsymbol{A}=\frac{b c \boldsymbol{c}}{e^{2}}$ for $\boldsymbol{e}$.
7. Solve $a=\frac{b}{\sqrt{1-\frac{d^{2}}{c^{2}}}}$ for $d$.
8. Solve $\boldsymbol{x}^{2}+y^{2}=\boldsymbol{z}^{2}$ for $\boldsymbol{y}$.
9. Solve $\boldsymbol{a}^{2} \boldsymbol{t}^{4}+\boldsymbol{b}^{2} \boldsymbol{t}^{2}-\boldsymbol{c}^{2}=\mathbf{0}$ for $\boldsymbol{t}$.

## STOP!

1. Solve $A=\pi \boldsymbol{R}^{2}$ for $\boldsymbol{R}$.

## Solution:

$$
\begin{aligned}
A & =\pi R^{2} \\
R^{2} & =\frac{A}{\pi} \\
R & = \pm \sqrt{\frac{A}{\pi}}
\end{aligned}
$$

2. Solve $\boldsymbol{E}=\boldsymbol{m} \boldsymbol{c}^{\mathbf{2}}$ for $\boldsymbol{c}$.

## Solution:

$$
\begin{aligned}
E & =m c^{2} \\
c^{2} & =\frac{E}{m} \\
c & = \pm \sqrt{\frac{E}{m}}
\end{aligned}
$$

3. Solve $S=2 \pi R^{2}+2 \pi R H$ for $R$.

## Solution:

$$
S=2 \pi R^{2}+2 \pi R H
$$

$2 \pi R^{2}+2 \pi R H-S=\mathbf{0}$ This is the quadratic formula with

$$
\begin{aligned}
& a=2 \pi, b=2 \pi H, c=-S \\
& R=\frac{-2 \pi H \pm \sqrt{(2 \pi H)^{2}-4(2 \pi)(-S)}}{2(2 \pi)} \\
&= \frac{-2 \pi H \pm \sqrt{4 \pi^{2} H^{2}+8 \pi S}}{4 \pi} \\
&= \frac{-2 \pi H \pm \sqrt{4\left(\pi^{2} H^{2}+2 \pi S\right)}}{4 \pi} \\
&= \frac{-2 \pi H \pm 2 \sqrt{\pi^{2} H^{2}+2 \pi S}}{4 \pi} \\
&= \frac{2\left(-\pi H \pm \sqrt{\pi^{2} H^{2}+2 \pi S}\right)}{4 \pi} \\
&= \frac{-\pi H \pm \sqrt{\pi^{2} H^{2}+2 \pi S}}{2 \pi}
\end{aligned}
$$

You could have used the special quadratic formula with $\overline{\boldsymbol{b}}=\boldsymbol{\pi} \boldsymbol{H}$.
4. Solve $(\boldsymbol{x}-\boldsymbol{H})^{2}+(\boldsymbol{y}-\boldsymbol{k})^{2}=\boldsymbol{R}^{2}$ for $\boldsymbol{y}$.

Solution:

$$
\begin{aligned}
(x-H)^{2}+(y-k)^{2} & =R^{2} \\
(y-k)^{2} & =R^{2}-(x-H)^{2} \\
y-k & = \pm \sqrt{R^{2}-(x-H)^{2}} \\
y & =k \pm \sqrt{R^{2}-(x-H)^{2}}
\end{aligned}
$$

5. Solve $\boldsymbol{A}=\boldsymbol{b} \boldsymbol{x}^{2} \boldsymbol{c}$ for $\boldsymbol{x}$.

## Solution:

$$
\begin{aligned}
A & =b x^{2} c \\
x^{2} & =\frac{A}{b c} \\
x & = \pm \sqrt{\frac{A}{b c}}
\end{aligned}
$$

6. Solve $\boldsymbol{A}=\frac{b c d}{e^{2}}$ for $\boldsymbol{e}$.

## Solution:

$$
\begin{aligned}
A & =\frac{b c d}{e^{2}} \\
e^{2} & =\frac{b c d}{A} \\
e & = \pm \sqrt{\frac{b c d}{A}}
\end{aligned}
$$

7. Solve $a=\frac{b}{\sqrt{1-\frac{d^{2}}{c^{2}}}}$ for $d$.

## Solution:

$$
\begin{aligned}
& a=\frac{b}{\sqrt{1-\frac{d^{2}}{c^{2}}}} \\
& a^{2}=\frac{b^{2}}{1-\frac{d^{2}}{c^{2}}} \\
& 1-\frac{d^{2}}{c^{2}}=\frac{b^{2}}{a^{2}} \\
& c^{2}-\frac{c^{2} d^{2}}{c^{2}}=\frac{c^{2} b^{2}}{a^{2}} \\
& c^{2}-d^{2}=\frac{c^{2} b^{2}}{a^{2}} \\
& c^{2}-\frac{c^{2} b^{2}}{a^{2}}=d^{2} \\
& d^{2}=c^{2}-\frac{c^{2} b^{2}}{a^{2}} \\
& d= \pm \sqrt{c^{2}-\frac{c^{2} b^{2}}{a^{2}}} \quad \text { This is correct and sufficient, } \\
& \text { but the last form below is more elegant. }
\end{aligned}
$$

$$
= \pm \sqrt{\frac{c^{2} a^{2}}{a^{2}}-\frac{c^{2} b^{2}}{a^{2}}}
$$

$$
= \pm \sqrt{\frac{c^{2}}{a^{2}}\left(a^{2}-b^{2}\right)}
$$

$$
= \pm \frac{c}{a} \sqrt{a^{2}-b^{2}}
$$

8. Solve $\boldsymbol{x}^{2}+\boldsymbol{y}^{2}=\boldsymbol{z}^{2}$ for $\boldsymbol{y}$.

## Solution:

$$
\begin{aligned}
x^{2}+y^{2} & =z^{2} \\
y^{2} & =z^{2}-x^{2} \\
y & = \pm \sqrt{z^{2}-x^{2}}
\end{aligned}
$$

9. Solve $a^{2} t^{4}+b^{2} t^{2}-c^{2}=0$ for $t$.

## Solution:

$$
a^{2} t^{4}+b^{2} t^{2}-c^{2}=0
$$

$$
a^{2}\left(t^{2}\right)^{2}+b^{2}\left(t^{2}\right)-c^{2}=\mathbf{0} \quad \text { Replace } t^{2} \text { by } x
$$

$$
a^{2} x^{2}+b^{2} x-c^{2}=0 \quad \text { This is the quadratic formula }
$$

$$
\begin{gathered}
a=a^{2}, b=b^{2}, c=-c^{2} \\
\frac{-b^{2} \pm \sqrt{\left(b^{2}\right)^{2}-4\left(a^{2}\right)\left(-c^{2}\right)}}{2 a^{2}} \\
t^{2}=\frac{-b^{2} \pm \sqrt{b^{4}+4 a^{2} c^{2}}}{2 a^{2}} \\
t= \pm \sqrt{\frac{-b^{2} \pm \sqrt{b^{4}+4 a^{2} c^{2}}}{2 a^{2}}}
\end{gathered}
$$

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## Chapter 72

## Introduction to the Arithmetic of Complex Numbers

### 72.1 Youtube

https://www.youtube.com/playlist?list=PLA82150A4BA98A6CA\&feature=view_all

### 72.2 Basics

Solve $\boldsymbol{x}^{2}+\mathbf{1}=\mathbf{0}$.
Then $x= \pm \sqrt{-1}$
But $\sqrt{-1}$ is not a solution we are used to. It is not what we call a "real" number. So why do we not expand the system of numbers we are used to. We are giving the expanded system the name imaginary numbers and refer to $\sqrt{\mathbf{- 1}}=\boldsymbol{i}$. We combine an imaginary number like $\mathbf{2 i}$ with a real number like 3 to create a complex number $\boldsymbol{z}=\mathbf{3 + 2 i}$.

Note that $2+3 i$ and $2-3 i$ are called complex conjugates.
A historical note: An imaginary number is a number whose square is negative. When any real number is squared, the result is never negative. Imaginary numbers have the form $\boldsymbol{b i}$ where $\boldsymbol{b}$ is a non-zero real number and $\boldsymbol{i}$ is the imaginary unit, defined such that $\boldsymbol{i}^{2}=\mathbf{- 1}$.

An imaginary number $\boldsymbol{b} \boldsymbol{i}$ can be added to a real number a to form a complex number of the form $\boldsymbol{a}+\boldsymbol{b} \boldsymbol{i}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are called, respectively, the "real part" and the "imaginary part" of the complex number. Imaginary numbers can therefore be thought of as complex numbers where the real part is zero, and vice versa.

The name "imaginary number" was originally coined in the 17th century. It was a derogatory term as such numbers were regarded by some as fictitious or useless, but today they have essential, concrete applications
in a variety of scientific, engineering, and related areas.
See http://en.wikipedia.org/wiki/Imaginary_number.
Real numbers are now a subset of complex numbers.
For example $\mathbf{5}=\mathbf{5}+\mathbf{0} \boldsymbol{i}$.
Since $i=\sqrt{-1}, i^{2}=-1$.

$$
\begin{aligned}
i & =\sqrt{-1} \\
i^{2} & =-1 \\
i^{3} & =i^{2} \cdot i=(-1) i=-i \\
i^{4} & =\left(i^{2}\right)^{2}=(-1)^{2}=1 \\
i^{5} & =i^{4} \cdot i=(1) i=\sqrt{-1} \\
i^{6} & =i^{4} \cdot i^{2}=(1)(-1)=-1 \\
i^{7} & =i^{4} \cdot i^{3}=(1) i^{3}=-i \\
i^{8} & =\left(i^{4}\right)^{2}=(1)^{2}=1 \\
i^{9} & =\left(i^{4}\right)^{2} \cdot i=(1) i=\sqrt{-1} \\
i^{10} & =\left(i^{4}\right)^{2} \cdot i^{2}=(1)(-1)=-1 \\
i^{11} & =\left(i^{4}\right)^{2} \cdot i^{3}=(1) i^{3}=-i \\
i^{12} & =\left(i^{4}\right)^{3}=(1)^{3}=1 \\
\cdots & =\cdots \\
i^{251} & =\left(i^{4}\right)^{62} \cdot i^{3}=(1) i^{3}=-i
\end{aligned}
$$

Note that in the last example you can divide 254 by $\mathbf{4}$ to get a remainder of $\mathbf{3}$. Thus $\boldsymbol{i}^{\mathbf{2 5 1}}=\boldsymbol{i}^{\mathbf{3}}$.
We will do arithmetic of complex numbers using $\boldsymbol{z}_{\mathbf{1}}=\mathbf{3}+\mathbf{5 i}$ and $\boldsymbol{z}_{\mathbf{2}}=\mathbf{4}-\mathbf{6 i}$.

### 72.3 Examples

Example 1:
Add $z_{1}=3+5 i$ to $z_{2}=4-6 i$.

## Solution:

$$
\begin{aligned}
z_{1}+z_{2} & =(3+5 i)+(4-6 i) \\
& =(3+4)+(5 i-6 i) \\
& =7+(5-6) i \\
& =7+(-1) i=7-i
\end{aligned}
$$

Example 2:
Subtract $z_{2}=4-6 i$ from $z_{1}=3+5 i$.

## Solution:

$$
\begin{aligned}
z_{1}-z_{2} & =(3+5 i)-(4-6 i) \\
& =3+5 i-4+6 i \\
& =(3-4)+(5 i+6 i) \\
& =-1+(5+6) i=-1+11 i
\end{aligned}
$$

Example 3:
Multiply $z_{1}=3+5 i$ and $z_{2}=4-6 i$.

## Solution:

$$
\begin{aligned}
z_{1} z_{2} & =(3+5 i)(4-6 i) \\
& =(3)(4-6 i)+(5 i)(4-6 i) \\
& =12-18 i+20 i-30 i^{2} \\
& =12+2 i-30(-1) \\
& =12+2 i+30=42+2 i
\end{aligned}
$$

Example 4:
Divide $\boldsymbol{z}_{1}=\mathbf{3}+5 \boldsymbol{i}$ by $\boldsymbol{z}_{2}=\mathbf{4 - 6 \boldsymbol { i }}$. Write your answer in the form $\boldsymbol{a}+\boldsymbol{b i}$ or $\boldsymbol{a}+\boldsymbol{i b}$.
Solution:

$$
\begin{aligned}
\frac{z_{1}}{z_{2}} & =\frac{3+5 i}{4-6 i} \\
& =\frac{3+5 i}{4-6 i} \cdot \frac{4+6 i}{4+6 i} \quad \text { Multiply numerator and denominator by the complex conjugate of the denominator. } \\
& =\frac{(3+5 i)(4+6 i)}{(4-6 i)(4+6 i)} \\
& =\frac{12+18 i+20 i+30 i^{2}}{(4)^{2}-(6 i)^{2}} \quad \text { Distribute in the numerator. In the } \\
& =\frac{12+38 i+30(-1)}{16-\left(36 i^{2}\right)} \\
& =\frac{12+38 i-30}{16-36(-1)} \\
& =\frac{-18+38 i}{16+36} \\
& =\frac{-18+38 i}{52}=\frac{-18}{52}+\frac{38}{52} i=-\frac{-9}{26}+\frac{19}{26} i \quad \text { After reducing. }
\end{aligned}
$$

### 72.4 Exercise 72

1. Add $z_{1}=7+8 i$ to $z_{2}=9-i$.
2. Subtract $z_{2}=\mathbf{9}-\boldsymbol{i}$ from $\boldsymbol{z}_{1}=\mathbf{7}+\mathbf{8 i}$.
3. Multiply $\boldsymbol{z}_{1}=3+5 i$ and $\boldsymbol{z}_{2}=4-6 \boldsymbol{i}$.
4. Divide $\boldsymbol{z}_{1}=\mathbf{7}+\mathbf{8 i}$ by $\boldsymbol{z}_{2}=\mathbf{9}-\boldsymbol{i}$. Write your answer in the form $\boldsymbol{a}+\boldsymbol{b} \boldsymbol{i}$ or $\boldsymbol{a}+\boldsymbol{i} \boldsymbol{b}$.
5. Simplify $\boldsymbol{i}^{\mathbf{4 9 4}}$.

## STOP!

1. Add $z_{1}=7+8 i$ to $z_{2}=9-i$.

## Solution:

$$
\begin{aligned}
z_{1}+z_{2} & =(7+8 i)+(9-i) \\
& =(7+9)+(8 i-i) \\
& =16+(8-1) i \\
& =16+7 i
\end{aligned}
$$

2. Subtract $\boldsymbol{z}_{\mathbf{2}}=\mathbf{9}-\boldsymbol{i}$ from $\boldsymbol{z}_{\mathbf{1}}=\mathbf{7}+\mathbf{8 i}$.

## Solution:

$$
\begin{aligned}
z_{1}-z_{2} & =(7+8 i)-(9-i) \\
& =(7-9)+(8 i+i) \\
& =-2+(8+1) i \\
& =-2+9 i
\end{aligned}
$$

3. Multiply $z_{1}=7+8 i$ and $z_{2}=9-i$.

## Solution:

$$
\begin{aligned}
z_{1} z_{2} & =(7+8 i)(9-i) \\
& =(7)(9-i)+(8 i)(9-i) \\
& =63-7 i+72 i-8 i^{2} \\
& =63+65 i-8(-1) \\
& =63+65 i+8 \\
& =71+65 i
\end{aligned}
$$

4. Divide $\boldsymbol{z}_{1}=\mathbf{7}+8 \boldsymbol{i}$ by $\boldsymbol{z}_{2}=\mathbf{9}-\boldsymbol{i}$. Write your answer in the form $\boldsymbol{a}+\boldsymbol{b} \boldsymbol{i}$ or $\boldsymbol{a}+\boldsymbol{i} \boldsymbol{b}$.

## Solution:

$$
\begin{aligned}
\frac{z_{1}}{z_{2}} & =\frac{7+8 i}{9-i} \\
& =\frac{7+8 i}{9-i} \cdot \frac{9+i}{9+i} \quad \text { Multiply numerator and denominator by the } \\
& =\frac{(7+8 i)(9+i)}{(9-i)(9+i)} \\
& =\frac{63+7 i+72 i+8 i^{2}}{(9)^{2}-(i)^{2}} \quad \text { Distribute in the numerator. In the conjugate of the denominator. } \\
& =\frac{63+79 i+8(-1)}{81-\left(i^{2}\right)} \\
& =\frac{63+79 i-8}{81-(-1)} \\
& =\frac{55+79 i}{81+1} \\
& =\frac{55+79 i}{82}=\frac{55}{82}+\frac{79}{82} i
\end{aligned}
$$

5. Simplify $\boldsymbol{i}^{\mathbf{4 9 4}}$.

Solution:
$494=492+2=4 \cdot 123+2$.
$i^{494}=\left(i^{4}\right)^{123} i^{2}$
$=(1)^{123}(-1)$
$=-1$

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## Chapter 73

## Graphs of Quadratic Equations

### 73.1 Youtube

https://www.youtube.com/playlist?list=PL81C5718A156F7732\&feature=view_all

### 73.2 Basics

We develop the basic graph for $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$, then move it around based on the given equation. Understand the step-by-step developments below. Do not generate a bunch of points in a table and connect the dots.

If the equation of the parabola is $\boldsymbol{y}=\boldsymbol{a}(\boldsymbol{x}-\boldsymbol{H})^{2}+\boldsymbol{K}$ then we know the vertex is at $(\boldsymbol{H}, \boldsymbol{K})$. It opens upward if $\boldsymbol{a}>\mathbf{0}$ and downward if $\boldsymbol{a}<\mathbf{0}$.

### 73.3 Examples

## Example 1:

Graph $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$.

## Solution:

First note that $\boldsymbol{x}^{\mathbf{2}} \geq \mathbf{0}$ so $\boldsymbol{y} \geq \mathbf{0}$. This means there is no curve below the $\boldsymbol{x}$-axis.
Then note that $(-x)^{2}=x^{2}$ which means that the parabola is symmetric with respect to (WRT) the $\boldsymbol{y}$-axis. (You can draw the graph in the the first quadrant and rotate it 180 degrees about the $\boldsymbol{y}$-axis.)

## Example 2:

Graph $\boldsymbol{y}=-\boldsymbol{x}^{2}$.

## Solution:

First note that $\boldsymbol{x}^{\mathbf{2}} \geq \mathbf{0}$ so $\boldsymbol{y} \leq \mathbf{0}$. This means there is no curve above the $\boldsymbol{x}$-axis.
Then note that $(-x)^{2}=x^{2}$ which means that the parabola is symmetric with respect to (WRT) the $\boldsymbol{y}$ axis. (You can draw the graph in the the first quadrant and rotate it 180 degrees about the $\boldsymbol{y}$-axis.)

## Example 3:

Graph $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$.

## Solution:

First note that $\boldsymbol{y}^{\mathbf{2}} \geq \mathbf{0}$ so $\boldsymbol{x} \geq \mathbf{0}$. This means there is no curve to the left of the $\boldsymbol{y}$-axis.
Then note that $(-\boldsymbol{y})^{\mathbf{2}}=\boldsymbol{y}^{\mathbf{2}}$ which means that the parabola is symmetric with respect to (WRT) the $\boldsymbol{x}$ axis. (You can draw the graph in the the first quadrant and rotate it 180 degrees about the $\boldsymbol{x}$-axis.)

## Example 4:

Graph $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}$.

## Solution:

First note that $\boldsymbol{y}^{\mathbf{2}} \geq \mathbf{0}$ so $\boldsymbol{x} \leq \mathbf{0}$. This means there is no curve to the riht of the $\boldsymbol{y}$-axis.
Then note that $(\boldsymbol{- y})^{2}=\boldsymbol{y}^{\mathbf{2}}$ which means that the parabola is symmetric with respect to (WRT) the $\boldsymbol{x}$ axis. (You can draw the graph in the the first quadrant and rotate it 180 degrees about the $\boldsymbol{x}$-axis.)

## Example 5:

Graph $y=(x-1)^{2}$.

## Solution:

Note that $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$ and $\boldsymbol{y}=(\boldsymbol{x}-\mathbf{1})^{\mathbf{2}}$ are exactly the same curve.
The vertex for $\boldsymbol{y}=\boldsymbol{x}^{2}$ occurs at the abscissa ( $\boldsymbol{x}$-value) at which the square term $\boldsymbol{x}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{x}=\mathbf{0}$.

The vertex for $\boldsymbol{y}=(\boldsymbol{x}-\mathbf{1})^{\mathbf{2}}$ occurs at the abscissa at which the square term $(\boldsymbol{x}-\mathbf{1})^{\mathbf{2}}$ is 0 which is $\boldsymbol{x}=\mathbf{1}$. The vertex is at $(\mathbf{1}, \mathbf{0})$.

The vertical line $\boldsymbol{x}=\mathbf{1}$ is the axis of symmetry.

## Example 6:

Graph $y=-(x+2)^{2}$.

## Solution:

First note the similarity between $\boldsymbol{y}=-\boldsymbol{x}^{2}$ and $\boldsymbol{y}=-(\boldsymbol{x}+\mathbf{2})^{\mathbf{2}}$. They are exactly the same curve.

The vertex for $\boldsymbol{y}=-\boldsymbol{x}^{2}$ occurs at the abscissa ( $\boldsymbol{x}$-value) at which the square term $\boldsymbol{x}^{2}$ is 0 , which is $\boldsymbol{x}=\mathbf{0}$.

The vertex for $\boldsymbol{y}=-(\boldsymbol{x}+\mathbf{2})^{\mathbf{2}}$ occurs at the abscissa ( $\boldsymbol{x}$-value) at which the square term $(\boldsymbol{x}+\mathbf{2})^{\mathbf{2}}$ is 0 which is $\boldsymbol{x}=\mathbf{- 2}$. The vertex is at $(-2,0)$.

The vertical line $\boldsymbol{x}=\mathbf{- 2}$ is the axis of symmetry.

## Example 7:

Graph $x=(y-3)^{2}$.

## Solution:

First note the similarity between $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$ and $\boldsymbol{x}=(\boldsymbol{y}-\mathbf{3})^{\mathbf{2}}$. They are exactly the same curve.

The vertex for $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$ occurs at the ordinate ( $\boldsymbol{y}$-value) at which the square term $\boldsymbol{y}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{y}=\mathbf{0}$.

The vertex for $\boldsymbol{x}=(\boldsymbol{y}-\mathbf{3})^{\mathbf{2}}$ occurs at the ordinate ( $\boldsymbol{y}$-value) at which the square term $(\boldsymbol{y}-\mathbf{3})^{2}$ is 0 which is $\boldsymbol{y}=\mathbf{3}$. The vertex is at $(0,3)$.

The horizontal line $\boldsymbol{y}=\mathbf{3}$ is the axis of symmetry.

## Example 8:

Graph $x=-(y+4)^{2}$.

## Solution:

First note the similarity between $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}$ and $x=-(y+4)^{2}$. They are exactly the same curve. The vertex for $\boldsymbol{x}=-\boldsymbol{y}^{2}$ occurs at the ordinate ( $\boldsymbol{y}$-value) at which the square term $\boldsymbol{y}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{y}=\mathbf{0}$.
The vertex for $\boldsymbol{x}=-(\boldsymbol{y}+4)^{2}$ occurs at the ordinate $\left(\boldsymbol{y}\right.$-value) at which the square term $(\boldsymbol{y}+4)^{2}$ is 0 which is $\boldsymbol{y}=-\mathbf{4}$. The vertex is at $(\mathbf{0},-\mathbf{4})$.
The horizontal line $\boldsymbol{y}=-4$ is the axis of symmetry.

## Example 9:

Graph $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}+\mathbf{1}$.

## Solution:

First note the similarity between $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$ and $\boldsymbol{y}=$ $\boldsymbol{x}^{2}+\mathbf{1}$. They are exactly the same curve.
The vertex for $\boldsymbol{y}=\boldsymbol{x}^{2}$ occurs at the abscissa ( $\boldsymbol{x}$ value) at which the square term $\boldsymbol{x}^{2}$ is 0 , which is $\boldsymbol{x}=0$.
Every ordinate ( $\boldsymbol{y}$-value) of $\boldsymbol{y}=\boldsymbol{x}^{2}+\mathbf{1}$ is that of $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$ with 1 added. In other words, the graph of $\boldsymbol{y}=\boldsymbol{x}^{2}+1$ is that of $\boldsymbol{y}=\boldsymbol{x}^{2}$ translated upward by 1 unit.

## Example 10:

Graph $y=2-x^{2}$.

## Solution:

First note the similarity between $\boldsymbol{y}=-\boldsymbol{x}^{\mathbf{2}}$ and $\boldsymbol{y}=$ $-x^{2}+2$. They are exactly the same curve.
The vertex for $\boldsymbol{y}=-\boldsymbol{x}^{2}$ occurs at the abscissa ( $\boldsymbol{x}$ value) at which the square term $\boldsymbol{x}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{x}=\mathbf{0}$. Every ordinate ( $\boldsymbol{y}$-value) of $\boldsymbol{y}=-\boldsymbol{x}^{2}+\mathbf{2}$ is that of $\boldsymbol{y}=-\boldsymbol{x}^{\mathbf{2}}$ with 2 added. The graph of $\boldsymbol{y}=-\boldsymbol{x}^{2}+\mathbf{2}$ is that of $\boldsymbol{y}=-\boldsymbol{x}^{2}$ translated upward by 2 units.

As a bonus, the $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=-\boldsymbol{x}^{\mathbf{2}}+\mathbf{2}=\mathbf{0}$ or $(-\sqrt{2}, \mathbf{0}),(\sqrt{\mathbf{2}}, \mathbf{0})$.

## Example 11:

Graph $\boldsymbol{x}=\boldsymbol{y}^{2}-3$.

## Solution:

First note the similarity between $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$ and $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}-\mathbf{3}$. They are exactly the same curve.

The vertex for $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$ occurs at the ordinate ( $\boldsymbol{y}$-value) at which the square term $\boldsymbol{y}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{y}=\mathbf{0}$.

Every abscissa ( $\boldsymbol{x}$-value) of $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}-\mathbf{3}$ is that of $\boldsymbol{x}=\boldsymbol{y}^{2}$ with 3 subtracted. In other words, the graph of $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}-\mathbf{3}$ is that of $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$ translated to the left by 3 units.

As a bonus, the $\boldsymbol{y}$-intercepts are obtained by setting $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}-\mathbf{3}=\mathbf{0}$ or $(\mathbf{0},-\sqrt{\mathbf{3}})$ and $(0, \sqrt{3})$.

## Example 12:

Graph $\boldsymbol{x}=-\boldsymbol{y}^{2}+\mathbf{5}$.

## Solution:

First note the similarity between $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}$ and $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}+\mathbf{5}$. They are exactly the same curve.

The vertex for $\boldsymbol{x}=-\boldsymbol{y}^{2}$ occurs at the ordinate ( $\boldsymbol{y}$-value) at which the square term $\boldsymbol{y}^{\mathbf{2}}$ is 0 , which is $\boldsymbol{y}=\mathbf{0}$.

Every abscissa ( $\boldsymbol{x}$-value) of $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}+\mathbf{5}$ is that of $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}$ with 5 added. In other words, the graph of $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}+\mathbf{5}$ is that of $\boldsymbol{x}=\boldsymbol{-} \boldsymbol{y}^{\mathbf{2}}$ translated to the right by 5 units.

As a bonus, the $\boldsymbol{y}$-intercepts are obtained by setting $\boldsymbol{x}=-\boldsymbol{y}^{2}+\mathbf{5}=\mathbf{0}$ to get $(\mathbf{0},-\sqrt{\mathbf{5}})$ and $(0, \sqrt{5})$.

## Example 13:

Graph $y=(x+2)^{2}+3$.

## Solution:

First plot $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}$, a parabola opening upward with vertex $(\mathbf{0}, \mathbf{0})$.

To plot $\boldsymbol{y}=(\boldsymbol{x}+\mathbf{2})^{\mathbf{2}}$, shift the parabola above 2 units to the left so that the vertex is $(\mathbf{- 2 , 0})$.

Finally to plot $\boldsymbol{y}=(\boldsymbol{x}+\mathbf{2})^{\mathbf{2}}+\mathbf{3}$ the previous parabola is translated upward 3 units. The vertex will be at $(-2,3)$.

As an added bonus, the $\boldsymbol{y}$-intercept is $\boldsymbol{y}=(\mathbf{0}+$ $2)^{2}+3=7$.

## Example 14:

Graph $y=-(x-1)^{2}+4$.

## Solution:

First plot $\boldsymbol{y}=-\boldsymbol{x}^{2}$, a parabola opening downward with vertex $(\mathbf{0}, \mathbf{0})$.

To plot $\boldsymbol{y}=-(\boldsymbol{x}-\mathbf{1})^{2}$, shift the parabola above 1 unit to the right so that the vertex is $(\mathbf{1}, \mathbf{0})$.

Finally to plot $y=-(x-1)^{2}+4$ the previous parabola is translated upward 4 units. The vertex will be at $(1,4)$.

As an added bonus, the $\boldsymbol{y}$-intercept is $y=-(0-1)^{2}+4=3$.

Also the $x$-intercepts are obtained by setting

$$
\begin{aligned}
y=-(x-1)^{2}+4 & =0 \\
(x-1)^{2} & =4 \\
x-1 & = \pm 2
\end{aligned}
$$

Thus $(\mathbf{3}, \mathbf{0})$ and $(-\mathbf{1}, \mathbf{0})$ are the $\boldsymbol{x}$-intercepts.

## Example 15:

Graph $x=(y-3)^{2}-4$.

## Solution:

First plot $\boldsymbol{x}=\boldsymbol{y}^{\mathbf{2}}$, a parabola opening to the right with vertex $(\mathbf{0}, \mathbf{0})$.
To plot $\boldsymbol{x}=(\boldsymbol{y}-\mathbf{3})^{\mathbf{2}}$, shift the parabola above 3 units upward so that the vertex is $(\mathbf{0}, \mathbf{3})$.
Finally to plot $x=(y-3)^{2}-4$ the previous parabola is translated 4 units to the left. The vertex will be at $(-4,3)$.
As an added bonus, the $\boldsymbol{x}$-intercept is $\boldsymbol{x}=(\mathbf{0}-\mathbf{3})^{\mathbf{2}}-$ $4=5$.
Also the $\boldsymbol{y}$ intercepts are obtained by setting $x=(y-3)^{2}-4=0$

$$
\begin{aligned}
(y-3)^{2} & =4 \\
y-3 & = \pm 2
\end{aligned}
$$

Thus $(\mathbf{0}, \mathbf{1})$ and $(\mathbf{0}, \mathbf{5})$ are the $\boldsymbol{y}$-intercepts.

## Example 16:

Graph $x=-(y+2)^{2}+5$.

## Solution:

First plot $\boldsymbol{x}=-\boldsymbol{y}^{\mathbf{2}}$, a parabola opening to the left with vertex $(\mathbf{0}, \mathbf{0})$.
To plot $\boldsymbol{x}=-(\boldsymbol{y}+\mathbf{2})^{\mathbf{2}}$, shift the parabola above 2 units downward so that the vertex is $(\mathbf{0}, \mathbf{- 2})$.
Finally to plot $x=-(y+2)^{2}+5$ the previous parabola is translated 5 units to the right. The vertex will be at $(5,-2)$.
As an added bonus, the $\boldsymbol{x}$-intercept is $\boldsymbol{x}=-(\mathbf{0}+$ $2)^{2}+5=1$.
Also the $\boldsymbol{y}$ intercepts are obtained by setting $x=-(y+2)^{2}+5=0$

$$
(y+2)^{2}=5
$$

$$
y+2= \pm \sqrt{5}
$$

$$
y=-2 \pm \sqrt{5}
$$

$(0,-2+\sqrt{5})$ and $(0,-2-\sqrt{5})$ are the $\boldsymbol{y}$-intercepts.

We see how to graph the parabola $\boldsymbol{y}=\boldsymbol{a}(\boldsymbol{x}-\boldsymbol{H})^{2}+\boldsymbol{K}$. The vertex is at $\boldsymbol{V}(\boldsymbol{H}, \boldsymbol{K})$, The parabola opens upward if $\boldsymbol{a}>\mathbf{0}$ and downward if $\boldsymbol{a}<\mathbf{0}$. When $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{y}=\boldsymbol{a} \boldsymbol{x}^{2}+\boldsymbol{b} \boldsymbol{x}+\boldsymbol{c}$ use completing the square to transform the equation. In general

$$
\begin{aligned}
y & =a x^{2}+b x+c \\
& =a\left(x^{2}+\frac{b}{a} x\right)+c \\
& =a\left(x^{2}+\frac{b}{a} x+\frac{b^{2}}{(2 a)^{2}}\right)-\frac{a b^{2}}{(2 a)^{2}}+c \\
& =a\left(x+\frac{b}{2 a}\right)^{2}-\frac{b^{2}}{4 a}+c \\
& =a(x-H)^{2}+K
\end{aligned}
$$

The vertex is at $\boldsymbol{V}\left(-\frac{b}{2 \boldsymbol{a}}, \boldsymbol{K}\right)$. Most students use $\boldsymbol{x}=-\frac{\boldsymbol{b}}{2 \boldsymbol{a}}, \boldsymbol{K}$ to find the abscissa of the vertex, then plug this value into $f(\boldsymbol{x})$ to find the ordinate of the vertex. I prefer completing the square in each case.

## Example 17:

Sketch (plot, graph) $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}+\mathbf{6 x}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## Solution:

$$
\begin{aligned}
y & =x^{2}+6 x \\
& =x^{2}+6 x+\left(\frac{6}{2}\right)^{2}-\left(\frac{6}{2}\right)^{2} \\
& =(x+3)^{2}-9
\end{aligned}
$$

The vertex is at $\boldsymbol{V}(\mathbf{- 3}, \mathbf{- 9})$
The $\boldsymbol{y}$-intercept is obtained by setting $\boldsymbol{x}=\mathbf{0}$, $\boldsymbol{y}=\mathbf{0}^{\mathbf{2}}+\mathbf{6}(\mathbf{0})=\mathbf{0}$, so $(\mathbf{0}, \mathbf{0})$ is the $\boldsymbol{y}$-intercept.

The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$. $\mathbf{0}=\boldsymbol{x}^{2}+\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{x}+\mathbf{6})$ or $(\mathbf{0}, \mathbf{0})$ and $(-6,0)$.

## Example 18:

Sketch (plot, graph) $\boldsymbol{y}=4 \boldsymbol{x}^{2}-3 \boldsymbol{x}-\frac{\mathbf{7}}{\mathbf{1 6}}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## Solution:

$$
\begin{aligned}
y & =4 x^{2}-3 x-\frac{7}{16} \\
& =4\left(x^{2}-\frac{3}{4} x\right)-\frac{7}{16} \\
& =4\left(x^{2}-\frac{3}{4} x+\frac{3^{2}}{8^{2}}\right)-\frac{4 \cdot 3^{2}}{8^{2}}-\frac{7}{16} \\
& =4\left(x-\frac{3}{8}\right)^{2}-\frac{9}{16}-\frac{7}{16} \\
& =4\left(x-\frac{3}{8}\right)^{2}-\frac{16}{16} \\
& =4\left(x-\frac{3}{8}\right)^{2}-1, \text { The vertex is at } V\left(\frac{3}{8},-1\right)
\end{aligned}
$$

The $y$-intercept is obtained by setting $x=0, y=4(0)^{2}-3(0)-\frac{7}{16}=-\frac{7}{16}$, so $\left(0,-\frac{7}{16}\right)$ is the $y$-intercept.

The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$.

$$
\begin{aligned}
4\left(x-\frac{3}{8}\right)^{2}-1 & =0 \\
4\left(x-\frac{3}{8}\right)^{2} & =1 \\
\left(x-\frac{3}{8}\right)^{2} & =\frac{1}{4} \\
x-\frac{3}{8} & = \pm \frac{1}{2} \\
x & =\frac{3}{8} \pm \frac{4}{8}, \text { thus }\left(\frac{7}{8}, 0\right) \text { and }\left(-\frac{1}{8}, 0\right)
\end{aligned}
$$

## Example 19:

Sketch (plot, graph) $\boldsymbol{y}=-\mathbf{2} \boldsymbol{x}^{2}+\mathbf{8 x}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## Solution:

$$
\begin{aligned}
y & =-2 x^{2}+8 x \\
& =-2\left(x^{2}-4 x+4\right)+8 \\
& =-2(x-2)^{2}+8
\end{aligned}
$$

The vertex is at $\boldsymbol{V}(\mathbf{2}, \mathbf{8})$
The $\boldsymbol{y}$-intercept is obtained by setting $\boldsymbol{x}=\mathbf{0}$, $y=-2(0)^{2}+8(0)=0$, so $(0,-0)$ is the $y$ intercept.
The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$.

$$
\begin{aligned}
0 & =-2(x-2)^{2}+8 \\
2(x-2)^{2} & =8 \\
(x-2)^{2} & =4 \\
x-2 & = \pm 2 \\
x & =2 \pm 2
\end{aligned}
$$

or $(0,0)$ and $(4,0)$.

### 73.4 Exercise 73

1. Sketch (plot, graph) $\boldsymbol{y}=\boldsymbol{x}^{2}-8 \boldsymbol{x}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.
2. Sketch $\boldsymbol{y}=\mathbf{6} \boldsymbol{x}^{\mathbf{2}}-\mathbf{5 x}+\frac{\mathbf{1}}{\mathbf{2 4}}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.
3. Sketch $\boldsymbol{y}=-\mathbf{5} \boldsymbol{x}^{\mathbf{2}}+\mathbf{3 0 \boldsymbol { x }}+\mathbf{8}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## STOP!

1. Sketch (plot, graph) $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}-\mathbf{8 x}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## Solution:

$$
\begin{aligned}
y & =x^{2}-8 x \\
& =x^{2}-8 x+\left(\frac{8}{2}\right)^{2}-\left(\frac{8}{2}\right)^{2} \\
& =(x-4)^{2}-16
\end{aligned}
$$

The vertex is at $\boldsymbol{V}(\mathbf{4}, \mathbf{- 1 6})$
The $\boldsymbol{y}$-intercept is obtained by setting $\boldsymbol{x}=\mathbf{0}$, $y=0^{2}-\mathbf{8}(0)=0$, so $(0,0)$ is the $y$-intercept.
The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$.
$0=x^{2}-8 x=x(x-8)$ or $(0,0)$ and $(8,0)$.
2. Sketch (plot, graph) $\boldsymbol{y}=\mathbf{6} \boldsymbol{x}^{2}-\mathbf{5 x}+\frac{\mathbf{1}}{\mathbf{2 4}}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.
Solution:

$$
\begin{array}{rlr} 
& = & 6 x^{2}-5 x-\frac{1}{24} \\
& = & 6\left(x^{2}-\frac{5}{6} x\right)+\frac{1}{24} \\
& = & 6\left(x^{2}-\frac{5}{6} x+\frac{5^{2}}{12^{2}}\right)-\frac{6 \cdot 5^{2}}{12^{2}}+\frac{1}{24} \\
& = & 6\left(x-\frac{5}{12}\right)^{2}-\frac{25}{24}+\frac{1}{24} \\
& = & 6\left(x-\frac{5}{12}\right)^{2}-\frac{24}{24} \\
& = & 6\left(x-\frac{5}{12}\right)^{2}-1 \\
=6\left(x-\frac{5}{12}\right)^{2}-1
\end{array}
$$

The vertex is at $\boldsymbol{V}\left(\frac{\mathbf{5}}{\mathbf{1 2}}, \mathbf{- 1}\right)$
The $\boldsymbol{y}$-intercept is obtained by setting $\boldsymbol{x}=\mathbf{0}$,
$y=6(0)^{2}-5(0)+\frac{1}{24}$, so $\left(0, \frac{1}{24}\right)$ is the $y$ -
intercept.
The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$.

$$
\begin{aligned}
6\left(x-\frac{5}{12}\right)^{2}-1 & =0 \\
6\left(x-\frac{5}{12}\right)^{2} & =1 \\
\left(x-\frac{5}{12}\right)^{2} & =\frac{1}{6} \\
x-\frac{5}{12} & = \pm \frac{1}{\sqrt{6}} \\
x & =\frac{5}{12} \pm \frac{\sqrt{6}}{6} \\
x & =\frac{5}{12} \pm \frac{2 \sqrt{6}}{12} \\
x & =\frac{5 \pm 2 \sqrt{6}}{12}
\end{aligned}
$$

or approximately $(\mathbf{0 . 8 2}, \mathbf{0})$ and $(\mathbf{0 . 1}, \mathbf{0})$.
3. Sketch (plot, graph) $\boldsymbol{y}=-\mathbf{5} \boldsymbol{x}^{\mathbf{2}}+\mathbf{3 0} \boldsymbol{x}+\mathbf{8}$. Indicate the vertex, the $\boldsymbol{y}$-intercept, and if possible, the $\boldsymbol{x}$-intercepts.

## Solution:

$$
\begin{aligned}
y & =-5 x^{2}+30 x+8 \\
& =-5\left(x^{2}-6 x+9\right)+45+8 \\
& =-5(x-3)^{2}+53
\end{aligned}
$$

The vertex is at $\boldsymbol{V}(\mathbf{3}, \mathbf{5 3})$
The $\boldsymbol{y}$-intercept is obtained by setting $\boldsymbol{x}=\mathbf{0}$, $y=-5(0)^{2} \mathbf{3 0}(0)+8=8$, so $(0,8)$ is the $y$ intercept.
The $\boldsymbol{x}$-intercepts are obtained by setting $\boldsymbol{y}=\mathbf{0}$.
$0=-5(x 33)^{2}+53$
$5(x-3)^{2}=53$
$(x-3)^{2}=\frac{53}{5}$
$x-3= \pm \sqrt{\frac{53}{5}}$
$x=3 \pm \sqrt{\frac{53}{5}}$
or approximately $(-\mathbf{0 . 3}, \mathbf{0})$ and $(6.3,0)$.

